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Modeling of EBW CD in spherical tokamaks

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Motivation

- For heating and current drive in high-density core plasmas of spherical tokamaks, **electromagnetic waves with electron cyclotron (EC) range of frequencies** have been extensively studied theoretically and experimentally.
- The propagation and absorption of EC waves are **usually analyzed by the ray tracing method** based on geometrical optics for waves with short wave length.
- In a plasma with high density or low magnetic field, however, **the presence of cutoff layer may** prevent the waves from penetrating into the central part from the low field side.
- In this case, **full wave analysis of EBW (Electron Bernstein Wave)** is required for evaluating the power absorption profile and **Fokker-Planck analysis of electron momentum distribution function** for evaluating the driven current profile.
- In the present analysis, the full wave analysis and the Fokker-Planck analysis of EBW using integral formulation are discussed.

Full Wave Analysis

- **Boundary-value problem of Maxwell's equation with fixed ω**
  - $\boldsymbol{E}$: wave electric field
  - $\overleftrightarrow{\epsilon}$: dielectric tensor

$$\nabla\times\nabla\times\boldsymbol{E} = \frac{\omega^2}{c^2}\overleftrightarrow{\epsilon}\cdot\boldsymbol{E} + \mathrm{i}\,\omega\mu_0\boldsymbol{j}_{\rm ext}$$

- **Merit of full wave analysis**
  - Wave length longer than the scale length of medium
  - Propagation over an evanescent layer
  - Coupling to antenna
  - Formation of standing wave

Finite Larmor Radius Effects in Full Wave Analysis

- **Fast wave approximation**:
  - Estimate $k_\perp\rho$ from fast wave $k_\perp$ in cold plasma approximation
  - Applicable parameter range is limited: fast wave, traveling wave
- **Differential operator approach**: $k_\perp\rho \to i\rho\partial/\partial r_\perp$
  - Expansion in $k_\perp\rho$: not applicable for $k_\perp\rho \gtrsim 1$
  - Difficult to cyclotron harmonics higher than the third order
- **Spectral approach**: Fourier transform in the inhomogeneous direction
  - This approach can be applied to the case $k_\perp\rho > 1$.
  - All the wave field spectra are coupled with each other.
  - Solving a dense matrix equation requires large computer resources.
  - **AORSA code (Jaeger, ORNL)**
- **Integral operators**: $\int \epsilon(x-x')\cdot E(x')\mathrm{d}x'$
  - This approach can be applied to the case $k_\perp\rho > 1$
  - Correlations are localized within several Larmor radii
  - Necessary to solve a large band matrix
  - **Sauter(NF, 1992)**, **TASK/W1**

Integral Formulation of Wave-Particle Interaction

- **General form of dielectric tensor**

$$\nabla\times\nabla\times\boldsymbol{E}(\boldsymbol{r},\omega) - \frac{\omega^2}{c^2}\int_V \mathrm{d}\boldsymbol{r}'\,\overleftrightarrow{\epsilon}(\boldsymbol{r},\boldsymbol{r}';\omega)\cdot\boldsymbol{E}(\boldsymbol{r}',\omega) - \mathrm{i}\,\omega\mu_0\boldsymbol{J}_{\rm ext}(\boldsymbol{r},\omega) = \boldsymbol{0}$$

- **Particle orbit**:

$$\boldsymbol{r} = \boldsymbol{r}' + \Delta\boldsymbol{r}(\boldsymbol{v},\boldsymbol{r},t-t')$$
$$\boldsymbol{v} = \boldsymbol{v}' + \Delta\boldsymbol{v}(\boldsymbol{v},\boldsymbol{r},t-t')$$

- **Perturbed distribution from Vlasov equation**:

$$f(\boldsymbol{r},\boldsymbol{v},t) = -\frac{q}{m}\int_{-\infty}^{t}\mathrm{d}t'\left[\boldsymbol{E}(\boldsymbol{r}') + \boldsymbol{v}'\times\boldsymbol{B}(\boldsymbol{r}')\right]\cdot\frac{\partial f_0(\boldsymbol{r}',\boldsymbol{v}')}{\partial\boldsymbol{v}'}\,\mathrm{e}^{-\mathrm{i}\,\omega t'}$$

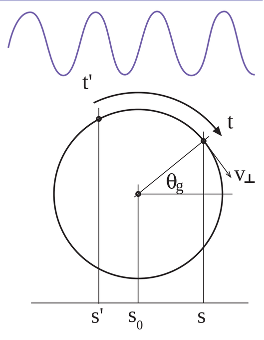
- **Induced current**:

$$\boldsymbol{j}(\boldsymbol{r}) = \int\mathrm{d}\boldsymbol{v}\,q\boldsymbol{v}f(\boldsymbol{r},\boldsymbol{v},t)\,\mathrm{e}^{\,\mathrm{i}\,\omega t} = \int\mathrm{d}\boldsymbol{r}'\,\overleftrightarrow{\sigma}(\boldsymbol{r}-\boldsymbol{r}',t-t')\cdot E(\boldsymbol{r}')$$

- **The integral form of the conductivity tensor** is defined by

$$\overleftrightarrow{\sigma}(\boldsymbol{r},\boldsymbol{r}',t-t') = -\frac{q}{m}\int_{-\infty}^{t}\mathrm{d}t'\,\frac{\partial f_0(\boldsymbol{r}',\boldsymbol{v}')}{\partial\boldsymbol{v}'}\cdot\left[\boldsymbol{v} + \frac{1}{\mathrm{i}\,\omega}\boldsymbol{v}\cdot\boldsymbol{v}'\times\nabla\times\right]\Bigg|_{\substack{\boldsymbol{r}'=\boldsymbol{r}-\Delta\boldsymbol{r}(\boldsymbol{v},\boldsymbol{r},t-t')\\ \boldsymbol{v}'=\boldsymbol{v}-\Delta\boldsymbol{v}(\boldsymbol{v},\boldsymbol{r},t-t')}}$$

Variable transformation



- **Transformation of integral variables**
  - Transformation from velocity space variables $(v_\perp,\theta_g)$ to particle position s' and guiding center position s_0
  - Jacobian: $J = \dfrac{\partial(v_\perp,\theta_g)}{\partial(s',s_0)} = -\dfrac{\omega_c^2}{v_\perp\sin\omega_c\tau}$.
- **Express $v_\perp$ and θ_g by the use of s', s_0, $\tau = t - t'$**

$$v_\perp^2 = \left(s_0 - \frac{s+s'}{2}\right)^2\frac{\omega_c^2}{\cos^2\frac{1}{2}\omega_c\tau} + \left(\frac{s-s'}{2}\right)^2\frac{\omega_c^2}{\sin^2\frac{1}{2}\omega_c\tau}$$

$$v_\perp\cos\theta_g = \omega_c\frac{s-s'}{2}\frac{1}{\tan\frac{1}{2}\omega_c\tau} + \omega_c\left(s_0 - \frac{s+s'}{2}\right)\tan\frac{1}{2}\omega_c\tau$$

$$v_\perp\sin\theta_g = \omega_c(s_0 - s)$$

- **Integral over τ**:
  - Fourier expansion with respect to periodic cyclotron motion

Equilibrium Velocity Distribution Function

- **For arbitrary velocity distribution function**
  - Numerical integration with respect to $v_\parallel$ and $\theta = \omega_c t$ is necessary
- **Anisotropic Maxwellian distribution**:
  - Perpendicular temperature: $T_\perp$, parallel temperature: $T_\parallel$

$$f_0(s_0,\boldsymbol{v}) = n_0\left(\frac{m}{2\pi T_\perp}\right)^{3/2}\left(\frac{T_\perp}{T_\parallel}\right)^{1/2}\exp\left[-\frac{v_\perp^2}{2v_{T\perp}^2} - \frac{v_\parallel^2}{2v_{T\parallel}^2}\right]$$

  - **Integral over $v_\parallel$**: Plasma dispersion function: $Z(\eta)$
  - **Integral over $\theta \equiv \omega_c t$**: Reduced to four types of kernel functions

Kernel Functions

- **Kernel Function and its integral**

$$F_\ell^{(i)}(X,Y) \equiv \frac{1}{2\pi^2}\int_0^\pi \mathrm{d}\theta\exp\left[-\frac{X^2}{1+\cos\theta} - \frac{Y^2}{1-\cos\theta}\right]f_\ell^{(i)}(\theta)$$

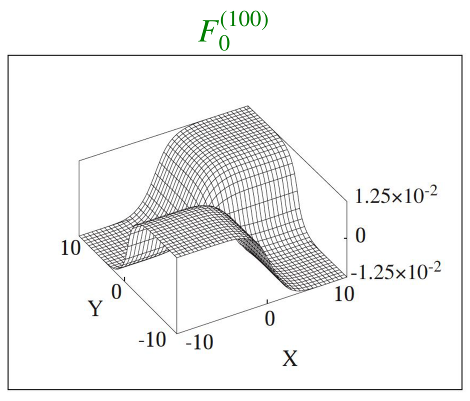
where

$$f_\ell^{(i)}(\theta) = \begin{cases} \dfrac{\cos\ell\theta}{\sin\theta} & (i=1)\\ \sin\ell\theta & (i=2)\\ \dfrac{\sin\ell\theta}{\sin^2\theta} & (i=3)\\ \dfrac{\cos\theta\sin\ell\theta}{\sin^2\theta} & (i=4)\end{cases}$$

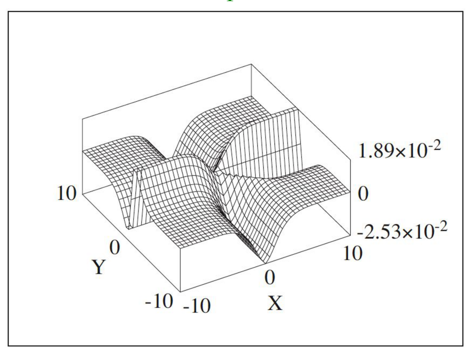
– Integral –

$$F_\ell^{(i,j,k)}(X,Y) \equiv \int_0^Y \mathrm{d}Y'\int_0^{X+Y'}\mathrm{d}X'X'^{j}\,Y'^{k}\,F_\ell^{(i)}(X',Y')$$

$F_0^{(100)}$



$F_1^{(100)}$



Final Form of Induced Current

- **Induced current**:

$$\cdot\begin{pmatrix}J_+^{mn}(s)\\ J_-^{mn}(s)\\ J_\parallel^{mn}(s)\end{pmatrix} = \int\mathrm{d}s'\sum_{m'n'}\overleftrightarrow{\sigma}^{m'n'mn}(s,s')\cdot\begin{pmatrix}E_+^{m'n'}(s')\\ E_-^{m'n'}(s')\\ E_\parallel^{m'n'}(s')\end{pmatrix}$$

- **Electrical conductivity**:

$$\overleftrightarrow{\sigma}^{m'n'mn}(s,s') = -in_0\frac{q^2}{m}\sum_\ell\int\mathrm{d}s_0\int_0^{2\pi}\mathrm{d}\chi_0\int_0^{2\pi}\mathrm{d}\zeta_0\exp i\left\{(m'-m)\chi_0 + (n'-n)\zeta_0\right\}\overleftrightarrow{H}_\ell(s,s',s_0,\chi_0,\zeta_0)$$

- **Matrix coefficients**: $\overleftrightarrow{H}_\ell(s,s',s_0,\chi_0,\zeta_0)$
  - Four kinds of **Kernel functions**
    - function of $s-s_0$, $s'-s_0$ and harmonics number ℓ
    - localized within several thermal Larmor radii
    - depending on guiding center position (s_0,χ_0,ζ_0)
  - **Plasma dispersion function**

Coefficient Matrix $\overleftrightarrow{H}_\ell$

$$H_{\ell xx} = -nA_1F_\ell^{(0)}2$$
$$H_{\ell yx} = \mathrm{i}A_1(X-Y)\left\{(X-Y)F_\ell^{(0)}3 - (X+Y)F_\ell^{(0)}4\right\}$$
$$H_{\ell zx} = -\mathrm{i}A_2\left\{(X-Y)F_\ell^{(0)}3 - (X+Y)F_\ell^{(0)}4\right\}$$
$$H_{\ell xy} = -\mathrm{i}A_1(X+Y)\left\{(X+Y)F_\ell^{(0)}3 - (X-Y)F_\ell^{(0)}4\right\}$$
$$H_{\ell yy} = -A_1(X+Y)(X-Y)F_\ell^{(0)}1$$
$$H_{\ell zy} = A_2(X+Y)F_\ell^{(0)}1$$
$$H_{\ell xz} = \mathrm{i}A_2\left\{(X+Y)F_\ell^{(0)}3 - (X-Y)F_\ell^{(0)}4\right\}$$
$$H_{\ell yz} = A_2(X-Y)F_\ell^{(0)}1$$
$$H_{\ell zz} = \frac{\sqrt{2}v_{T\parallel}\eta_\ell}{v_{T\perp}}A_2F_\ell^{(0)}1$$

where the kernel functions

$$F_\ell^{(i)}(X,Y) \equiv \frac{1}{2\pi^2}\int_0^\pi\mathrm{d}\theta\exp\left[-\frac{X^2}{1+\cos\theta} - \frac{Y^2}{1-\cos\theta}\right]f_\ell^{(i)}(\theta)$$

with

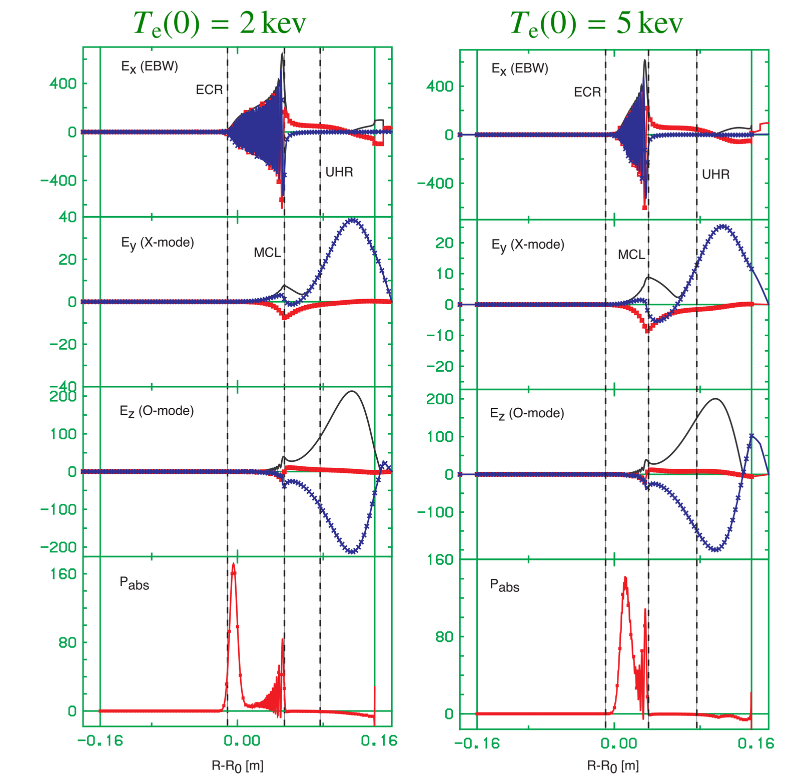
$$X \equiv \frac{\Omega}{v_{T\perp}}\left(x_0 - \frac{x+x'}{2}\right),\qquad Y \equiv \frac{\Omega}{2v_\perp}(x-x'),\qquad \eta \equiv \frac{\omega - n\Omega}{k_\parallel v_{T_\parallel}\sqrt{2}}.$$

$$A_1 \equiv \frac{\omega}{\sqrt{2}k_\parallel v_{T\parallel}}Z(\eta_\ell) + \left(1 - \frac{T_\perp}{T_\parallel}\right)\frac{Z'(\eta_\ell)}{2},\qquad A_2 \equiv \frac{\omega}{2k_\parallel v_{T\perp}}\left\{\frac{T_\perp}{T_\parallel} + \ell\frac{\Omega}{\omega}\left(1 - \frac{T_\perp}{T_\parallel}\right)\right\}Z'(\eta_\ell).$$

One-Dimensional Analysis

O-X-B excitation

major radius	$R_0 = 0.22\,\mathrm{m}$
minor radius	$a = 0.15\,\mathrm{m}$
central magnetic field	$B_0 = 0.08\,\mathrm{T}$
toroidal mode number	$n_\phi = 24$
central electron density	$3\times10^{17}\,\mathrm{m}^{-3}$



Evolution of Momentum Distribution Function

- **Full wave analysis for arbitrary velocity distribution function**
  - **Dielectric tensor**:

$$\nabla\times\nabla\times\boldsymbol{E}(\boldsymbol{r}) - \frac{\omega^2}{c^2}\int\mathrm{d}\boldsymbol{r}_0\int\mathrm{d}\boldsymbol{r}'\,\frac{\boldsymbol{p}'}{m\gamma}\frac{\partial f_0(\boldsymbol{p}',\boldsymbol{r}_0)}{\partial\boldsymbol{p}'}\cdot\boldsymbol{K}_1(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}_0)\cdot\boldsymbol{E}(\boldsymbol{r}') = \mathrm{i}\,\omega\mu_0\boldsymbol{j}_{\rm ext}$$

  where $\boldsymbol{r}_0$ is the gyrocenter position.

- **Fokker-Planck analysis including finite Larmor radius effects**
  - **Quasi-linear operator**

$$\frac{\partial f_0}{\partial t} + \left(\frac{\partial f_0}{\partial\boldsymbol{p}}\right)_{\boldsymbol{E}} + \frac{\partial}{\partial\boldsymbol{p}}\int\mathrm{d}\boldsymbol{r}\int\mathrm{d}\boldsymbol{r}'\boldsymbol{E}(\boldsymbol{r})\,\boldsymbol{E}(\boldsymbol{r}')\cdot\boldsymbol{K}_2(\boldsymbol{r},\boldsymbol{r}',\boldsymbol{r}_0)\cdot\frac{\partial f_0(\boldsymbol{p}',\boldsymbol{r}_0,t)}{\partial\boldsymbol{p}'} = \left(\frac{\partial f_0}{\partial\boldsymbol{p}}\right)_{\rm col}$$

- The kernels $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are closely related and localized in the region $|\boldsymbol{r}-\boldsymbol{r}_0| \lesssim 3\rho$ and $|\boldsymbol{r}'-\boldsymbol{r}_0| \lesssim 3\rho$.

Consideration on Quasi-Linear Diffusion Coefficient

- **Ordering**

$$\boldsymbol{E}(\boldsymbol{r},t) = \varepsilon\boldsymbol{E}_1(\boldsymbol{r},t) + \varepsilon^2\boldsymbol{E}_2(\boldsymbol{r},t) + \cdots$$
$$\boldsymbol{B}(\boldsymbol{r},t) = \varepsilon\boldsymbol{B}_1(\boldsymbol{r},t) + \varepsilon^2\boldsymbol{B}_2(\boldsymbol{r},t) + \cdots$$
$$f(\boldsymbol{v},\boldsymbol{r},t) = f_0(\boldsymbol{v},\varepsilon^2t) + \varepsilon f_1(\boldsymbol{v},\boldsymbol{r},t) + \varepsilon^2 f_2(\boldsymbol{v},\boldsymbol{r},t) + \cdots$$

- **Vlasov equation**

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{r}} + \frac{q_s}{m_s}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}} = 0$$

- **First-order distribution function**

$$f_1(\boldsymbol{r},\boldsymbol{v},t) = -\frac{q_s}{m_s}\int_{-\infty}^{t}\mathrm{d}t'\left(\boldsymbol{E}_1(\boldsymbol{r}',\boldsymbol{v}',t') + \boldsymbol{v}'\times\boldsymbol{B}_1(\boldsymbol{r}',\boldsymbol{v}',t')\right)\cdot\frac{\partial f_0}{\partial\boldsymbol{v}'}$$

- **Second-order Vlasov equation**

$$\frac{\partial f_0}{\partial t} + \frac{\partial f_2}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f_2}{\partial\boldsymbol{r}} + \frac{q_s}{m_s}(\boldsymbol{E}_1 + \boldsymbol{v}\times\boldsymbol{B}_1)\cdot\frac{\partial f_1}{\partial\boldsymbol{v}} + \frac{q_s}{m_s}(\boldsymbol{E}_2 + \boldsymbol{v}\times\boldsymbol{B}_2)\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} = 0$$

- **Particle motion in a local orthogonal coordinates**

$$\boldsymbol{v}(t) = \left(v_\perp\cos\theta, v_\perp\sin\theta, v_\parallel\right)$$
$$\boldsymbol{v}(t') = \left(v_\perp\cos[\theta + \Omega(t'-t)], v_\perp\sin[\theta + \Omega(t'-t)], v_\parallel\right)$$
$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \left(-\frac{v_\perp}{\Omega}\sin\theta, \frac{v_\perp}{\Omega}\cos\theta, 0\right)$$
$$\boldsymbol{r}(t') = \boldsymbol{r}_0 + \left(-\frac{v_\perp}{\Omega}\sin[\theta + \Omega(t'-t)], \frac{v_\perp}{\Omega}\cos[\theta + \Omega(t'-t)], 0\right)$$

- **Diffusion term**

$$\frac{\partial}{\partial\boldsymbol{v}}D(v_\parallel,v_\perp,\theta,\boldsymbol{r},t)\cdot\frac{\partial}{\partial\boldsymbol{v}}f_0(v_\parallel,v_\perp,\boldsymbol{r}_0) = -\frac{q_s^2}{m_s^2}\left[\boldsymbol{E}_1(\boldsymbol{r},t) + \boldsymbol{v}(t)\times\boldsymbol{B}_1(\boldsymbol{r},t)\right]\cdot\frac{\partial}{\partial\boldsymbol{v}} \times \int_{-\infty}^{t}\mathrm{d}t'\left[\boldsymbol{E}_1(\boldsymbol{r}',t') + \boldsymbol{v}(t')\times\boldsymbol{B}_1(\boldsymbol{r}',t')\right]\cdot\frac{\partial}{\partial\boldsymbol{v}'}f_0(v_\parallel,v_\perp,\boldsymbol{r}_0)$$

- **Quasi-linear diffusion coefficient**

$$D_{\rm QL}(v_\parallel,v_\perp,\boldsymbol{r}_0) = \frac{1}{2\pi/\Omega}\int_0^{2\pi/\Omega}\mathrm{d}t\int_0^{2\pi}\mathrm{d}\theta D(v_\parallel,v_\perp,\theta,\boldsymbol{r},t)$$

- **Magnetic toroidal coordinates**: (s,χ,ζ)
- **Variable transformation**: $(\theta,t) \longrightarrow (s,s')$

$$s = s_0 - \frac{v_\perp}{\Omega}\sin\theta,\qquad s' = s_0 - \frac{v_\perp}{\Omega}\sin[\theta + \Omega(t'-t)]$$

- **Jacobian**

$$J = \frac{\partial(\theta,t)}{\partial(s,s')} = \begin{pmatrix} -\dfrac{1}{v_\perp\cos\theta} & 0\\ 0 & -\dfrac{\Omega}{v_\perp\cos[\theta + \Omega(t'-t)]}\end{pmatrix} = \frac{\Omega}{v_\perp^2}\frac{\mathrm{sign}(p-p_0)}{\sqrt{1-\Omega^2(s-s_0)^2/v_\perp^2}}\frac{\mathrm{sign}(p'-p_0)}{\sqrt{1-\Omega^2(s'-s_0)^2/v_\perp^2}}$$

- **Poloidal and toroidal mode number**: (m,n)

$$\boldsymbol{E}(\boldsymbol{r},t) = \sum_{m,n}\boldsymbol{E}^{m,n}(s)\exp[\,\mathrm{i}\,(m\chi + n\zeta - \omega t)]$$

- **Quasi-linear diffusion coefficient**

$$D_{\rm QL}(v_\parallel,v_\perp,\boldsymbol{r}_0) = -\frac{2\pi q_s^2}{\Omega m_s^2}\int_0^{2\pi/\Omega}\mathrm{d}t\int_0^{2\pi}\mathrm{d}\theta\int_{-\infty}^{t}\mathrm{d}t' \times \left[\boldsymbol{E}_1(\boldsymbol{r},t) + \boldsymbol{v}(t)\times\boldsymbol{B}_1(\boldsymbol{r},t)\right]\left[\boldsymbol{E}_1(\boldsymbol{r}',t') + \boldsymbol{v}(t')\times\boldsymbol{B}_1(\boldsymbol{r}',t')\right]$$

$$= -\frac{2\pi q_s^2}{\Omega m_s^2}\int_{s_0-v_\perp/\Omega}^{s_0+v_\perp/\Omega}\mathrm{d}s\int_{s_0-v_\perp/\Omega}^{s_0+v_\perp/\Omega}\mathrm{d}s'\int_0^\infty\mathrm{d}\tau\,J\sum_{m,n} \times \boldsymbol{E}^{m,n*}(s)\cdot\overleftrightarrow{K}(v_\perp,v_\parallel,s-s_0,\chi_0,\zeta_0,0) \times \boldsymbol{E}^{m,n}(s')\cdot\overleftrightarrow{K}(v_\perp,v_\parallel,s'-s_0,\chi_0,\zeta_0,\tau)$$

Summary

- For the analysis of heating and current drive by the electron Bernstein waves, **integral formulation of full wave analysis and Fokker-Planck analysis** in an inhomogeneous plasma has been developed.
- Implementation of **integral form of dielectric tensor** for one-dimensional full wave analysis has been done for TASK/W1. The O-X-B mode conversion of EC waves were successfully described.
- **Future work**
  - Completion of integral form of quasi-linear velocity diffusion coefficients.
  - Two-dimensional full wave analysis including finite Larmor radius effects