

# Two-dimensional transport modeling in tokamak plasmas

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## Outline

- Background and Motivation
- Modeling of Two-dimensional Transport
- Present status of 2D Transport Code TASK/T2
- Summary

- **The core and peripheral plasmas are strongly coupled with each other in tokamaks.** The particle and heat fluxes from the core determine the behavior of the peripheral plasma, while the latter determines the edge density and temperature, boundary conditions of the core plasma.
- **The transport in the core and the peripheral regions have been analyzed separately until recently owing to the difference of modeling configurations.**
  - **In the core region of tokamaks**
    - By the use of flux-surface averaging, transport is **1D problem**.
    - A standard transport modeling is based on the **neoclassical transport theory** and **turbulent transport theory**
  - **In the peripheral region of tokamaks**
    - By the use of simplified transport models, transport is **2D problem**.
    - A standard transport modeling is based on the **Braginskii's equations** and **turbulent transport theory**

- **Integrated core-peripheral transport simulations by 1.5D core transport code and 2D peripheral transport code**
  - Quantities obtained by a core transport simulation lack poloidal dependence in the edge region.
  - In the case of H-mode plasmas, Braginskii's equations are not suitable in the edge region, since the plasma temperature becomes a few keV and the plasma becomes weakly collisional.
- **For more consistent core-peripheral transport simulation**
  - Two-dimensional transport modeling based on the neoclassical transport theory and turbulent transport theory applicable to both core and peripheral region are desirable.

We have formulated an axisymmetric two-dimensional transport model applicable to both the core and peripheral regions and are developing a two-dimensional transport code **TASK/T2**.

- **Assumptions**

- **Two-dimensional MHD equilibrium**

- Spatial variation of quantities are two-dimensional

- **Relaxation processes much slower than the Alfvén time scale**

- Time dependence of basis vector is negligible in the transport time scale

- **Radial force balance in the transport time scale**

- **Coordinates**

- **Magnetic surface coordinate (MSC):**  $(\rho, \chi, \zeta)$

- **Axisymmetric magnetic field in MSC:**  $B = \nabla\zeta \times \nabla\psi + I\nabla\zeta$

- **Transport oriented coordinate (TOC):**  $(\rho, \parallel, \zeta)$

We employ **MSC** to express **spatial variations of quantities**  
and **TOC** to express **components of vector quantities**  
for compatibility with neoclassical (NC) transport and peripheral transport theory

- Multi-fluid equations**

$$\begin{aligned}
 \left. \frac{\partial n_a}{\partial t} \right|_{\mathbf{x}} + \nabla \cdot \mathbf{\Gamma}_a &= S_{na} \\
 \left. \frac{\partial}{\partial t} (m_a n_a \mathbf{u}_a) \right|_{\mathbf{x}} + \nabla \cdot \overleftrightarrow{P}_a &= e_a n_a (\mathbf{E} + \mathbf{u}_a \times \mathbf{B}) + \mathbf{F}_a + \mathbf{S}_{ma} \\
 \left. \frac{\partial}{\partial t} \left( \frac{3}{2} p_a + \frac{1}{2} m_a n_a u_a^2 \right) \right|_{\mathbf{x}} + \nabla \cdot \mathbf{Q}_a &= e_a n_a \mathbf{E} \cdot \mathbf{u}_a + (\mathbf{F}_a \cdot \mathbf{u}_a + Q_{\Delta a}) + S_{pa} \\
 \left. \frac{\partial \mathbf{Q}_a}{\partial t} \right|_{\mathbf{x}} + \nabla \cdot \overleftrightarrow{R}_a &= \mathbf{F}_{qa}^{\text{Lor}} + \mathbf{G}_a + \mathbf{S}_{qa}
 \end{aligned}$$

where  $\mathbf{F}_{qa}^{\text{Lor}}$  is the Energy weighted (EW) Lorentz force defined as

$$\mathbf{F}_{qa}^{\text{Lor}} \equiv \frac{e_a}{m_a} \left[ \left( \frac{5}{2} p_a \overleftrightarrow{I} + \overleftrightarrow{\pi}_a \right) \cdot \mathbf{E} + \mathbf{Q}_a \times \mathbf{B} \right]$$

- Definition of higher moment quantities in each equation**

- Particle flux:  $\mathbf{\Gamma}_a \equiv n_a \mathbf{u}_a$
- Total stress tensor:  $\overleftrightarrow{P}_a \equiv m_a n_a \mathbf{u}_a \mathbf{u}_a + p_a \overleftrightarrow{I} + \overleftrightarrow{\pi}_a$
- Total heat flux:  $\mathbf{Q}_a \equiv \mathbf{q}_a + \frac{5}{2} p_a \mathbf{u}_a + \overleftrightarrow{\pi}_a \cdot \mathbf{u}_a + \frac{1}{2} m_a n_a u_a^2 \mathbf{u}_a$
- EW total stress tensor:  $\overleftrightarrow{R}_a \equiv \frac{5}{2} \frac{T_a}{m_a} p_a \overleftrightarrow{I} + \overleftrightarrow{r}_a + \mathbf{Q}_a \mathbf{u}_a + \mathbf{u}_a \mathbf{Q}_a - \frac{3}{2} p_a \mathbf{u}_a \mathbf{u}_a$

- **Collision terms**

- **Friction and Heat friction force:** 
$$\begin{bmatrix} \mathbf{F}_a \\ \mathbf{H}_a \end{bmatrix} \equiv \sum_b \begin{bmatrix} l_{11}^{ab} & -l_{12}^{ab} \\ -l_{21}^{ab} & l_{22}^{ab} \end{bmatrix} \begin{bmatrix} \mathbf{u}_b \\ \frac{2\mathbf{q}_b}{5p_b} \end{bmatrix}$$

- **Energy equipartition:** 
$$Q_{\Delta a} = \sum_b \frac{3}{2} n_a \frac{T_b - T_a}{\tau_{ab}}$$

- **EW friction force:** 
$$\mathbf{G}_a \equiv \frac{T_a}{m_a} \left( \frac{5}{2} \mathbf{F}_a + \mathbf{H}_a \right)$$

- **Viscosity tensors:** Only parallel viscosity tensors are taken into account

- **Parallel viscosity and heat viscosity tensor**

$$\begin{bmatrix} \overleftrightarrow{\pi}_{\parallel a} \\ \overleftrightarrow{\theta}_{\parallel a} \end{bmatrix} \equiv -\frac{3}{2} \begin{bmatrix} \mu_{1a} & \mu_{2a} \\ \mu_{2a} & \mu_{3a} \end{bmatrix} \begin{bmatrix} w_{ua} \\ w_{qa} \end{bmatrix} \left( \mathbf{e}_{\parallel} \mathbf{e}_{\parallel} - \frac{1}{3} \overleftrightarrow{I} \right)$$

–  $w_{ua} \equiv 2 \left( \nabla_{\parallel} u_{a\parallel} - \mathbf{u}_a \cdot \boldsymbol{\kappa} \right)$  and  $w_{qa} \equiv 2 \left[ \nabla_{\parallel} \left( \frac{2q_{a\parallel}}{5p_a} \right) - \frac{2\mathbf{q}_a}{5p_a} \cdot \boldsymbol{\kappa} \right]$

are quantities related to rate-of-strain tensor,

where  $\boldsymbol{\kappa} \equiv \mathbf{e}_{\parallel} \cdot \nabla \mathbf{e}_{\parallel}$  is magnetic curvature

- **EW parallel viscosity tensor:** 
$$\overleftrightarrow{r}_{a\parallel} \equiv \frac{T_a}{m_a} \left( \frac{5}{2} \overleftrightarrow{\pi}_{\parallel a} + \overleftrightarrow{\theta}_{\parallel a} \right)$$

- **These expressions are equivalent to that of the Hirshman's moment approach in the limit of equilibrium return flows**

- **Equation for particle transport**

$$\left. \frac{\partial n_a}{\partial t} \right|_{\mathbf{x}} + \nabla \cdot (n_a \mathbf{u}_a) = S_{na}$$

- **Equation for Force balance in radial direction**

$$\nabla \rho \cdot \nabla p_a = e_a n_a E^\rho + e_a \frac{IB}{\psi'} n_a u_{a\parallel} - e_a \frac{B^2}{\psi'} n_a u_{a\zeta}$$

- Since the time derivative of radial momentum is  $\mathcal{O}(\delta^3)$ , the lowest order radial force balance  $\mathcal{O}(\delta^0)$  is assumed for simplicity as previously indicated.

- **Equation for parallel momentum transport**

$$\left. \frac{\partial}{\partial t} (m_a n_a u_{a\parallel} B) \right|_{\mathbf{x}} + F_{ua\parallel}^{\text{ine}} B + B \nabla_{\parallel} p_a + F_{ua\parallel}^{\text{vis}} B = e_a n_a E_{\parallel} B + F_{a\parallel} B + S_{ma\parallel} B$$

- **Inertial force in parallel direction**

$$F_{ua\parallel}^{\text{ine}} B = \mathbf{B} \cdot \nabla \cdot (m_a n_a \mathbf{u}_{a\parallel} \mathbf{u}_{a\parallel}) = B \nabla_{\parallel} (m_a n_a u_{a\parallel} u_{a\parallel}) - m_a n_a u_{a\parallel} u_{a\parallel} \nabla_{\parallel} B$$

- **Viscous force in parallel direction:**

$$F_{ua\parallel}^{\text{vis}} B = \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_{a\parallel} = \frac{2}{3} B \nabla_{\parallel} \pi_{a\parallel} - \pi_{a\parallel} \nabla_{\parallel} B$$

- **Equation for toroidal momentum transport**

$$\left. \frac{\partial}{\partial t} (m_a n_a u_{a\zeta}) \right|_{\mathbf{x}} + F_{ua\zeta}^{\text{ine}} + F_{ua\zeta}^{\text{vis}} = e_a n_a E_\zeta + e_a n_a \psi' u_a^\rho + F_{a\zeta} + S_{ma\zeta}$$

- **Inertial force in the toroidal direction**

$$F_{ua\zeta}^{\text{ine}} = \nabla \cdot (m_a n_a u_{a\zeta} \mathbf{u}_a)$$

- **Viscous force in the toroidal direction**

$$\begin{aligned} F_{ua\zeta}^{\text{vis}} &= \nabla \cdot \left( R^2 \nabla \zeta \cdot \overleftrightarrow{\pi}_{a\parallel} \right) = \nabla \cdot \left[ \frac{I \pi_{a\parallel}}{B^2} B^\chi e_\chi + \pi_{\parallel a} \left( \frac{I^2}{B^2 R^2} - \frac{1}{3} \right) R^2 \nabla \zeta \right] \\ &= B \nabla_{\parallel} \left( \frac{I \pi_{a\parallel}}{B^2} \right) \end{aligned}$$

- **Equation for internal energy transport**

$$\left. \frac{3}{2} \frac{\partial p_a}{\partial t} \right|_{\mathbf{x}} + \nabla \cdot \left( \mathbf{Q}_a - \frac{1}{2} m_a n_a u_a^2 \mathbf{u}_a \right) = \mathbf{u}_a \cdot \nabla p_a + Q_a^{\text{vis}} + Q_{\Delta a} + S_{pa}$$

where  $S_{pa} \equiv S_{Ea} - \mathbf{S}_{ma} \cdot \mathbf{u}_a + \frac{1}{2} m_a u_a^2 S_{na}$  is internal energy source

- **Viscous heating due to parallel viscosity**

$$Q_a^{\text{vis}} \equiv \mathbf{u}_a \cdot \nabla \cdot \overleftrightarrow{\pi}_{a\parallel} = B \nabla_{\parallel} \left( \frac{u_{a\parallel} \pi_{\parallel a}}{B} \right) - \frac{1}{2} u_{ua} \pi_{\parallel a} - \frac{1}{3} \mathbf{u}_a \cdot \nabla \pi_{\parallel a}$$



- **Equation for EW force balance in radial direction**

$$\nabla \rho \cdot \nabla \left( \frac{5T_a}{2m_a} p_a \right) = \frac{e_a}{m_a} \left( \frac{5}{2} p_a E^\rho + \frac{IB}{\psi'} Q_{a\parallel} + \frac{B^2}{\psi'} Q_{a\zeta} \right)$$

- **Equation for parallel total heat flux transport**

$$\frac{\partial}{\partial t} (Q_{a\parallel} B) \Big|_{\mathbf{x}} + F_{qa\parallel}^{\text{ine}} B + B \nabla_{\parallel} \left( \frac{5}{2} \frac{T_a p_a}{m_a} \right) + F_{qa\parallel}^{\text{vis}} B = \frac{e_a}{m_a} \left( \frac{5}{2} p_a + \frac{2}{3} \pi_{\parallel a} \right) E_{\parallel} B + G_{a\parallel} B + S_{qa\parallel} B$$

where the EW inertial and viscous force in parallel direction are defined as

$$F_{qa\parallel}^{\text{ine}} B \equiv B \nabla_{\parallel} \left( Q_{a\parallel} u_{a\parallel} + u_{a\parallel} Q_{a\parallel} - \frac{3}{2} p_a u_{a\parallel} u_{a\parallel} \right) - \left( Q_{a\parallel} u_{a\parallel} + u_{a\parallel} Q_{a\parallel} - \frac{3}{2} p_a u_{a\parallel} u_{a\parallel} \right) \nabla_{\parallel} B$$

$$F_{qa\parallel}^{\text{vis}} B \equiv r_{\parallel a} \nabla_{\parallel} B + \frac{2}{3} B \nabla_{\parallel} r_{\parallel a}$$

- **Equation for toroidal total heat flux transport**

$$\frac{\partial Q_{a\zeta}}{\partial t} \Big|_{\mathbf{x}} + F_{qa\zeta}^{\text{ine}} + F_{qa\zeta}^{\text{vis}} = \frac{e_a}{m_a} \left[ \left( \frac{5}{2} p_a - \frac{1}{3} \pi_{\parallel a} \right) E_{\zeta} + \frac{I \pi_{\parallel a}}{B} E_{\parallel} + \psi' Q_a^\rho \right] + G_{a\zeta} + S_{qa\zeta}$$

where the EW inertial and viscous force in toroidal direction are defined as

$$F_{qa\zeta}^{\text{ine}} \equiv \nabla \cdot \left( Q_{a\zeta} \mathbf{u}_a + u_{a\zeta} \mathbf{Q}_a - \frac{3}{2} p_a u_{a\zeta} \mathbf{u}_a \right)$$

$$F_{qa\zeta}^{\text{vis}} \equiv B \nabla_{\parallel} \left( \frac{I r_{\parallel a}}{B^2} \right)$$

- Flux averaged parallel force and EW force balance up to  $\mathcal{O}(\delta)$

$$\left\langle \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_{\parallel a} \right\rangle = e_a n_a \langle E_{\parallel} B \rangle + \langle F_{a\parallel} B \rangle$$

$$\left\langle \mathbf{B} \cdot \nabla \cdot \overleftrightarrow{r}_{\parallel a} \right\rangle = \frac{5T_a}{2m_a} e_a n_a \langle E_{\parallel} B \rangle + \langle G_{a\parallel} B \rangle$$

- Equilibrium return flows inside the last closed flux surface

$$\bar{\mathbf{u}}_a = \omega_{ua} R^2 \nabla \zeta + L_{ua} \mathbf{B} \quad \text{and} \quad \bar{\mathbf{q}}_a = \omega_{qa} R^2 \nabla \zeta + L_{qa} \mathbf{B},$$

- $\omega_{ua}$  and  $\omega_{qa}$  are the toroidal angular frequencies
- $L_{ua}$  and  $L_{qa}$  are the quantities related to poloidal flows.

- $\omega_{ua}$  and  $\omega_{qa}$  with equilibrium return flows

$$\bar{\omega}_{ua} = 2(\nabla_{\parallel} B) L_{ua} \quad \text{and} \quad \bar{\omega}_{qa} = 2(\nabla_{\parallel} B) \frac{2L_{qa}}{5p_a}$$

- Parallel viscosity tensors with equilibrium return flows

$$\begin{bmatrix} \overleftrightarrow{\pi}_{\parallel a} \\ \overleftrightarrow{\theta}_{\parallel a} \end{bmatrix} \equiv -3 (\nabla_{\parallel} B) \begin{bmatrix} \mu_{1a} & \mu_{2a} \\ \mu_{2a} & \mu_{3a} \end{bmatrix} \begin{bmatrix} L_{ua} \\ \frac{2L_{qa}}{5p_a} \end{bmatrix} \left( \mathbf{e}_{\parallel} \mathbf{e}_{\parallel} - \frac{1}{3} \overleftrightarrow{I} \right)$$

- **Flux averaged parallel flow in equilibrium return flow limit**

$$\begin{aligned}\langle u_{a\parallel} B \rangle &= V_{1a} + L_{ua} \langle B^2 \rangle, & V_{1a} &= \frac{I}{B} \omega_{ua} \\ \langle q_{a\parallel} B \rangle &= \frac{5}{2} p_a V_{2a} + L_{qa} \langle B^2 \rangle, & V_{2a} &= \frac{I}{B} \omega_{qa}\end{aligned}$$

- **Matrix equation for poloidal rotations in our transport model**

$$\langle 3(\nabla_{\parallel} B)^2 \rangle \begin{bmatrix} \mu_{1a} & \mu_{2a} \\ \mu_{2a} & \mu_{3a} \end{bmatrix} \begin{bmatrix} L_{ua} \\ \frac{2L_{qa}}{5p_a} \end{bmatrix} = \sum_b \begin{bmatrix} l_{11}^{ab} & -l_{12}^{ab} \\ -l_{21}^{ab} & l_{22}^{ab} \end{bmatrix} \begin{bmatrix} V_{1b} B + L_{ub} \langle B^2 \rangle \\ V_{2b} B + \frac{2L_{qb}}{5p_b} \langle B^2 \rangle \end{bmatrix} + \begin{bmatrix} e_a n_a \langle E_{\parallel} B \rangle \\ 0 \end{bmatrix}$$

- This expression is equivalent to matrix equation for poloidal rotations in the conventional neoclassical transport theory

- **Evolution equation for  $B^\chi$ :** ( $B^\chi = \sqrt{g}d\psi/d\rho$ )

$$\left. \frac{\partial B^\chi}{\partial t} \right|_{\mathbf{x}} - \frac{1}{\sqrt{g}} \frac{\partial E_\rho}{\partial \rho} = 0$$

- **Evolution equation for  $B_\zeta$ :** ( $B_\zeta = I$ )

$$\left. \frac{\partial B_\zeta}{\partial t} \right|_{\mathbf{x}} + \frac{R^2}{\sqrt{g}} \left( \frac{\partial E_\chi}{\partial \rho} - \frac{\partial E_\rho}{\partial \xi_\chi} \right) = 0$$

- **Gauss's law**

$$\nabla \cdot \mathbf{E} = \sum_a \frac{e_a}{\epsilon_0}$$

- **Evolution equation for  $E_\chi$**

$$\frac{1}{c^2} \left. \frac{\partial E_\chi}{\partial t} \right|_{\mathbf{x}} + \frac{g_{\chi\chi}}{\sqrt{g}} \frac{\partial B_\zeta}{\partial \rho} + \mu_0 \sum_a \frac{e_a n_a u_{a\parallel} B - e_a n_a u_a^\zeta B_\zeta}{B^\chi} = 0$$

- **Evolution equation for  $E_\zeta$**

$$\frac{1}{c^2} \left. \frac{\partial E_\zeta}{\partial t} \right|_{\mathbf{x}} + R^2 \nabla \cdot \left( \frac{B^\chi}{\sqrt{g}R^2} \nabla \rho \right) + \mu_0 \sum_a e_a n_a u_{a\zeta} = 0$$

- **Time evolution algorithm**

1. **Grid construction step**

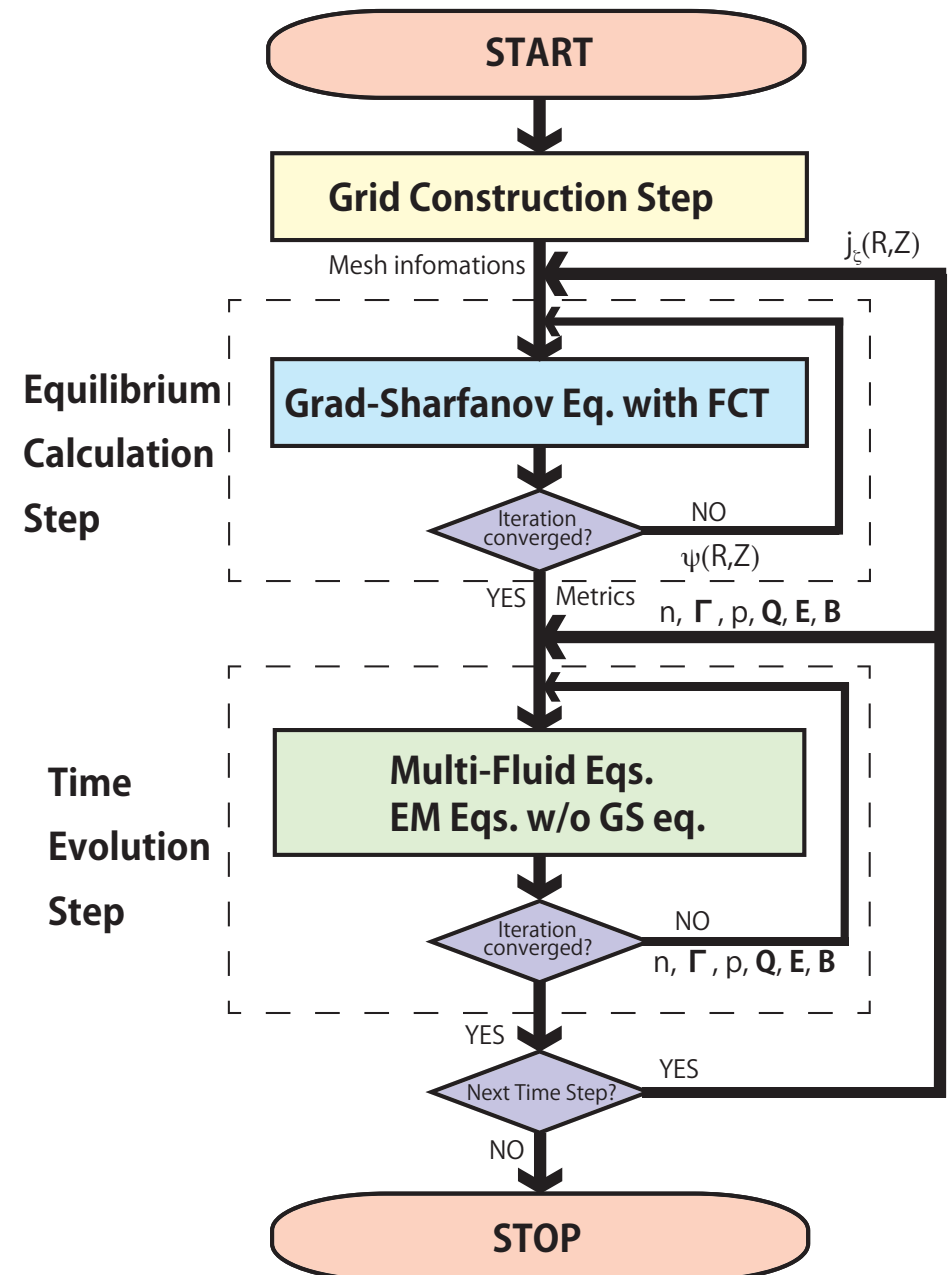
- TASK/T2
- Grid generation in MSC

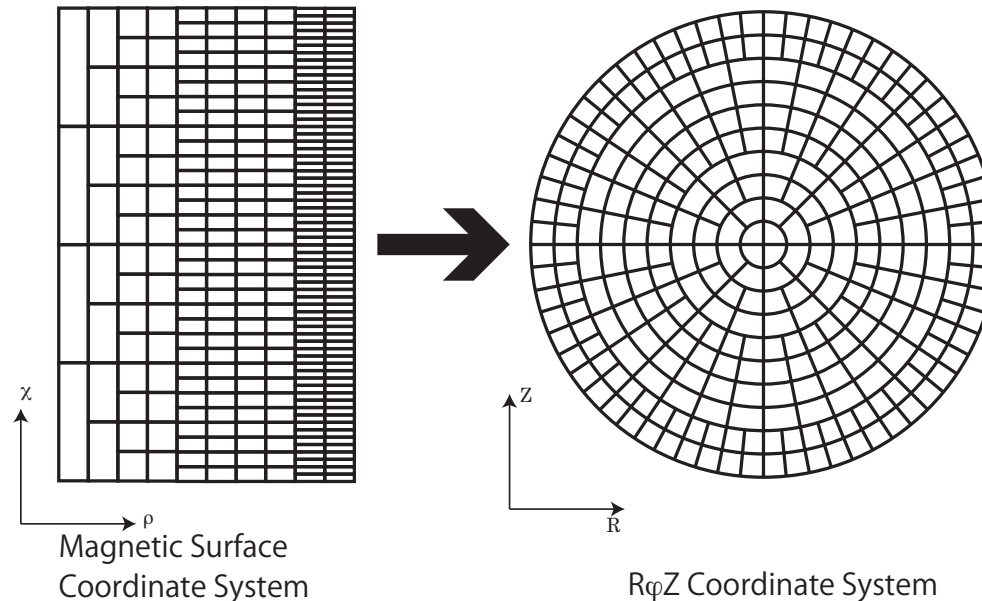
2. **Equilibrium calculation step**

- TASK/EQU
- Grad-Shafranov Eq.
- Metrics calculation

3. **Time evolution step**

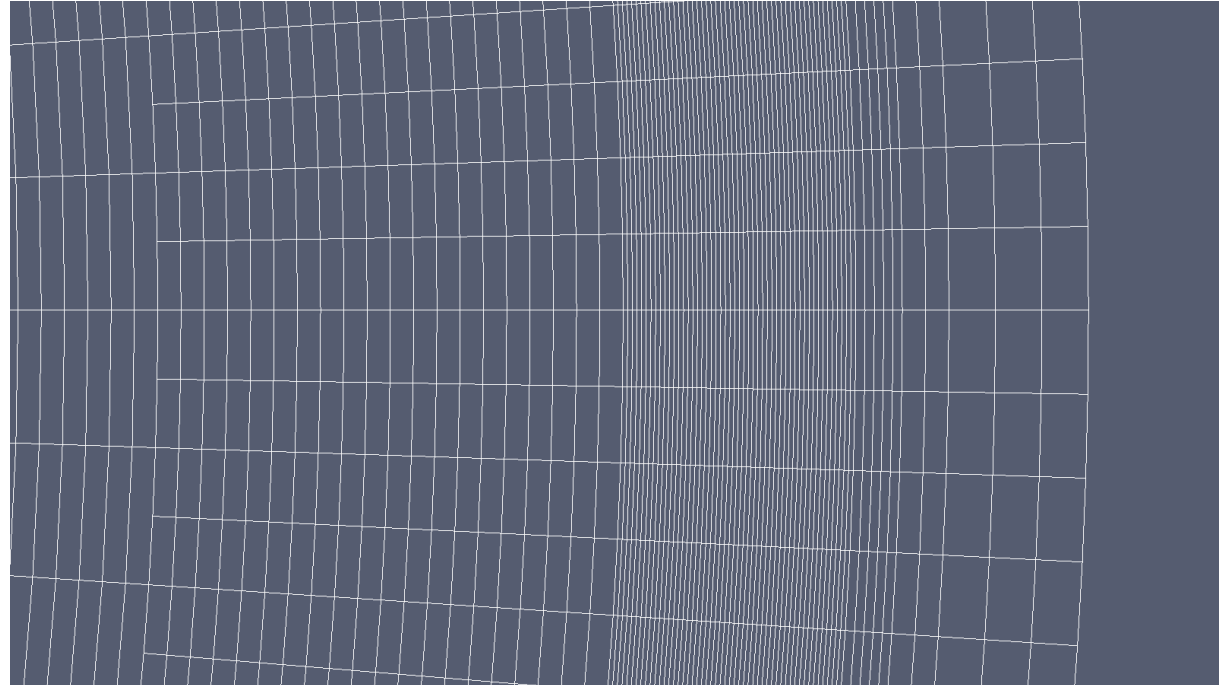
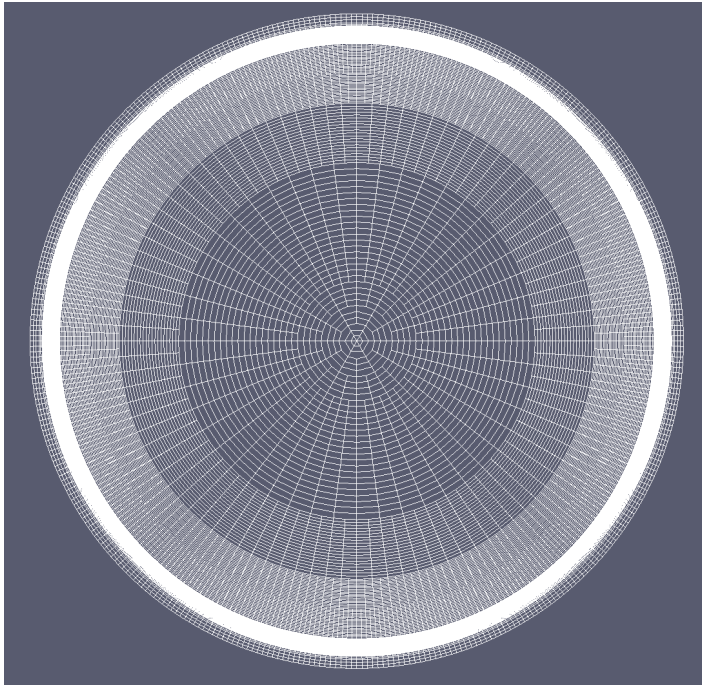
- TASK/T2, TASK/MTXP
- Multi-fluid Eqs., EM Eqs.
- Calculation of time evolution of plasma





- **Desirable grid property for two-dimensional transport analysis**
  - **Good separation of the parallel and perpendicular fluxes**
    - Rectangular grid whose sides are parallel or perpendicular to axes of MSC
  - **Uniform poloidal resolution in real space**
    - Hierarchical structure that becomes gradually finer at greater  $\rho$  region
  - **High flexibility of radial grid width**
    - Radial grid width can be easily changed if grid is structural

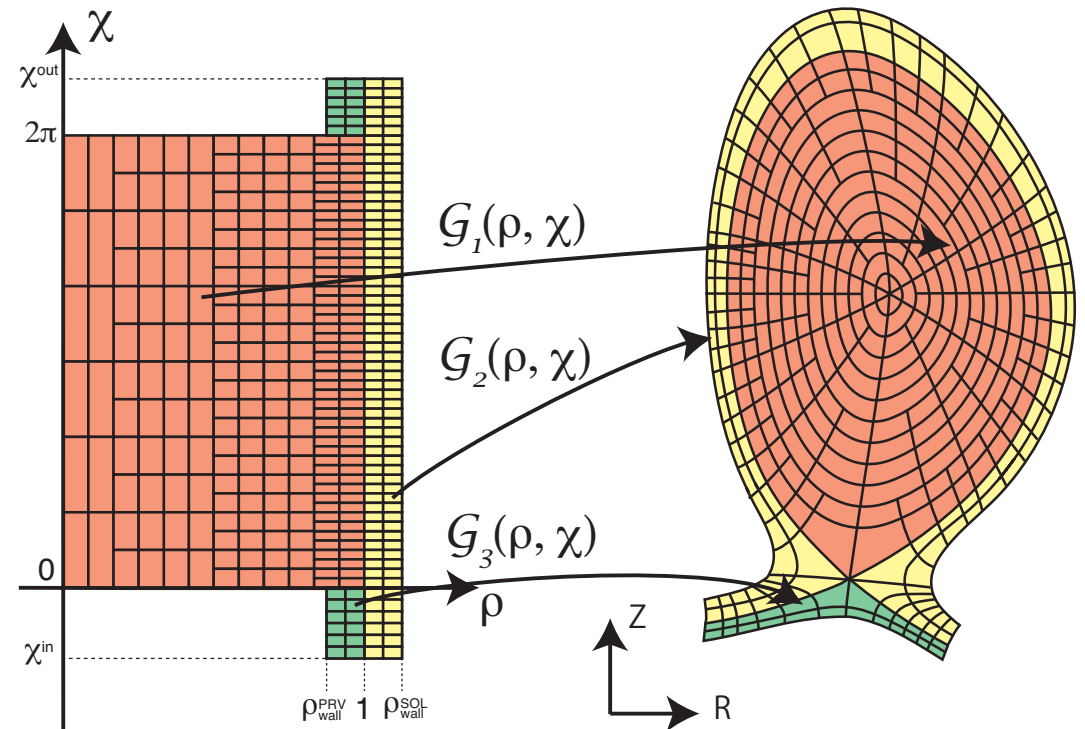
**Hierarchical rectangular grid in MSC is employed in TASK/T2**



- **Example of grid construction for limiter configuration**
  - **Hierarchical grid generation in cylindrical coordinate**
  - **Number of elements: 37590**
    - **Core region** ( $0.0 \leq r \leq 1.0$ ):  
Number of radial partitions: 100, Number of poloidal partitions: 6-384
    - **Peripheral region** ( $1.0 \leq r \leq 1.1$ ):  
Number of radial partitions: 50, Number of poloidal partitions: 384

- **Sub-domains of Single-null grid**

- **Core domain:** ■  
 $\rho \in [0, 1], \chi \in [0, 2\pi]$
- **SOL domain:** ■  
 $\rho \in [1, \rho_{\text{wall}}^{\text{SOL}}], \chi \in [\chi^{\text{in}}, \chi^{\text{out}}]$
- **Private domain:** ■  
 $\rho \in [\rho_{\text{wall}}^{\text{PRV}}, 1], \chi \in [\chi^{\text{in}}, 0]$   
 $\rho \in [\rho_{\text{wall}}^{\text{PRV}}, 1], \chi \in [2\pi, \chi^{\text{out}}]$



- **Each sub-domain has different mapping function from MSC to  $R\phi Z$  system**
- **Continuity between sub-domains at separatrix ( $\rho = 1, \chi^{\text{in}} \leq \chi \leq \chi^{\text{out}}$ )**
  - $0 \leq \chi \leq 2\pi$ : core and SOL domain
  - $\chi^{\text{in}} \leq \chi \leq 0$  and  $2\pi \leq \chi \leq \chi^{\text{out}}$ : SOL and private domain
- **Periodic conditions ( $0 \leq \rho \leq 1, \chi = 0, 2\pi$ )**
  - $0 \leq \rho \leq 1, \chi = +0, 2\pi - 0$ : for core domain
  - $\rho_{\text{wall}}^{\text{PRV}} \leq \rho \leq 1, \chi = -0, 2\pi + 0$ : for private domain



- **Governing equation:** Simultaneous Advection-Diffusion equation
- **Discretization scheme:** Finite Element Method
  - **Stabilization scheme:** SUPG-FEM
  - **Element type:** Structured bi-linear rectangular element
- **Time-advancing scheme:** Full implicit
- **Nonlinear calculation scheme:** Picard iteration
- **Matrix solver:** TASK/MTXP
  - **Parallel solver:** PETSc (Iterative method), MUMPS (direct method)
  - **Serial solver:** Gauss elimination for band matrix
- **Visualization:** Paraview

- **Summary**

- A set of equations required for 2D transport modeling of tokamak plasmas has been derived.
  - 2D transport equations have been derived from multi-fluid equations with neoclassical viscosity in MSC.
  - The neoclassical parallel viscosity and heat viscosity have been extended in order to be applicable in the open field region outside the last closed flux surface.

- **Future works**

- Developing the two-dimensional transport code TASK/T2
- Modeling of 2D momentum and heat flux transport due to the turbulent electric field based on quasi-linear transport theory
- 2D core and peripheral transport analysis of tokamaks with limiter configurations as a preliminary step to divertor configurations