Nonconforming Vector Finite Element Method for Fully-implicit Resistive MHD Simulations

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Background

MHD simulation for fusion plasmas requires the algorithm to ensure that it satisfies the divergence constraint on the vector (magnetic and velocity) fields,

- $\nabla \cdot \boldsymbol{b} = 0$
- $\nabla \cdot \boldsymbol{v} \sim 0$ (below the marginal stability limit)

Standard (i.e. "conforming") finite element method (FEM) does not satisfy those divergence-free constraints, and therefore it often generates unphysical spurious modes which interact with physical modes.

- ► We developed a novel FEM algorithm to ensure that vector variables in the MHD equations satisfy the divergence-free and curl-free constraints exactly in general coordinate systems.
- ► The formulation was implemented in a single-fluid resistive MHD code.

Standard (i.e. "Conforming") Finite Elements

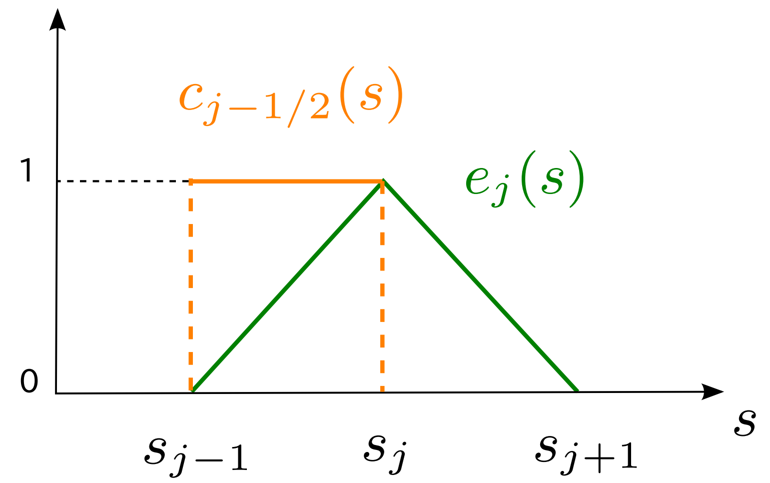
Periodic cylinder model : (s, θ, ϕ)

Fourier expanded : $\sum_{mn} f_{mn}(s)\,\exp(im\theta - in\phi)$

Contravariant vector :

$$B^\mu \equiv \sqrt{g} b^\mu, \quad \mu = s, \theta, \phi$$

Finite element



Conforming finite elements

$$\begin{pmatrix} B^s(s) \\ B^\theta(s) \\ B^\phi(s) \end{pmatrix} = \sum_j \begin{pmatrix} B^s_j \\ B^\theta_j \\ B^\phi_j \end{pmatrix} \underline{e_j(s)}$$

A single basis function

$$\sqrt{g}\,\nabla \cdot \boldsymbol{b} = \sum_j \frac{B^s_j - B^s_{j-1}}{s_j - s_{j-1}} \underline{c_{j-1/2}(s)} + \sum_j (imB^\theta_j - inB^\phi_j)\underline{e_j(s)} \neq 0$$

Mixture of basis functions

Standard finite element solution does not satisfy the divergence-free condition.

Nonconforming Vector Finite Element Formulation

We introduce an idea like 'Nonconforming' that different types of basis functions are used for a contravariant and a covariant vector.

$$A^s(s) = \sum_j A^s_j e_j(s), \quad \begin{pmatrix} A^\theta(s) \\ A^\phi(s) \end{pmatrix} = \sum_j \begin{pmatrix} A^\theta_{j-1/2} \\ A^\phi_{j-1/2} \end{pmatrix} c_{j-1/2}(s)$$

$$a_s(s) = \sum_j a_{s,j-1/2} c_{j-1/2}(s), \quad \begin{pmatrix} a_\theta(s) \\ a_\phi(s) \end{pmatrix} = \sum_j \begin{pmatrix} a_{\theta,j} \\ a_{\phi,j} \end{pmatrix} e_j(s)$$

The divergence and curl of a vector field

$$\sqrt{g}\,\nabla\cdot\boldsymbol{a} = \sum_j \left(\frac{A^s_j - A^s_{j-1}}{s_j - s_{j-1}} + imA^\theta_{j-1/2} - inA^\phi_{j-1/2} \right) c_{j-1/2}(s)$$

$$\sqrt{g}(\nabla\times\boldsymbol{a})^s = \sum_j (ima_{\phi,j} + ina_{\theta,j}) e_j(s)$$

$$\sqrt{g}(\nabla\times\boldsymbol{a})^\theta = \sum_j \left((-in)a_{s,j-1/2} - \frac{a_{\phi,j} - a_{\phi,j-1}}{s_j - s_{j-1}} \right) c_{j-1/2}(s)$$

$$\sqrt{g}(\nabla\times\boldsymbol{a})^\phi = \sum_j \left((-im)a_{s,j-1/2} + \frac{a_{\theta,j} - a_{\theta,j-1}}{s_j - s_{j-1}} \right) c_{j-1/2}(s)$$

A single basis function

Significant Feature of Nonconforming Vector FEM

- Inside each element,

$$\nabla \cdot (\nabla \times \boldsymbol{a}) \equiv 0$$

$\nabla \times \nabla f \equiv 0$ if the scalar function $f(s)$ is defined as $f(s) = \sum_j f_j e_j(s)$

Nonconforming finite element solution guarantees a divergence-free field has no divergence error and a curl-free field has no curl error.

- The covariant/contravariant metric transformation is NOT given in the discrete sense by the local metric tensor, as defined

$$a^\mu = g^{\mu\nu} a_\nu, \quad a_\mu = g_{\mu\nu} a^\nu$$

$$\Downarrow$$

Here, we introduce one more idea that the equation of the covariant/contravariant metric transformation is substituted into the weak form.

Weak Formulation of the Metric Transformation

Covariant vector : $\bar{\boldsymbol{a}}\ (\bar{\boldsymbol{w}})$ / Contravariant vector : $\boldsymbol{a}\ (\boldsymbol{w})$

† We denote the covariant vector with a bar and the contravariant vector without a bar

'Norm conserving condition' is imposed

$$\langle \bar{\boldsymbol{w}}, \boldsymbol{a} \rangle = \langle \bar{\boldsymbol{w}}, \bar{\boldsymbol{a}} \rangle, \quad i.e., \quad \int \sqrt{g} w_\mu a^\mu ds = \int \sqrt{g} g^{\mu\nu} w_\mu a_\nu ds$$

$$\mu,\ \nu = s,\ \theta,\ \phi$$

or

$$\langle \boldsymbol{w}, \bar{\boldsymbol{a}} \rangle = \langle \boldsymbol{w}, \boldsymbol{a} \rangle, \quad i.e., \quad \int \sqrt{g} w^\mu a_\mu ds = \int \sqrt{g} g_{\mu\nu} w^\mu a^\nu ds$$

e.g. :

Condition : $\langle \bar{\boldsymbol{w}}, \boldsymbol{a} \rangle = \langle \bar{\boldsymbol{w}}, \bar{\boldsymbol{a}} \rangle$, Cylinder model

$$\left[\int c_{j-1/2}(s) e_k(s) ds \right]_{jk} A^s_k = \left[\int_{s_{j-1}}^{s_j} \sqrt{g} g^{ss} ds \right]_{jj} a_{s,j-1/2}$$

$$\left[\int e_j(s) c_{k-1/2}(s) ds \right]_{jk} A^\theta_{k-1/2} = \left[\int \sqrt{g} g^{\theta\theta} e_j(s) e_k(s) ds \right]_{jk} a_{\theta,k}$$

$$\left[\int e_j(s) c_{k-1/2}(s) ds \right]_{jk} A^\phi_{k-1/2} = \left[\int \sqrt{g} g^{\phi\phi} e_j(s) e_k(s) ds \right]_{jk} a_{\phi,k}$$

Application to an Eigenvalue Problem

Magnetic diffusion equation (An eigenvalue problem)

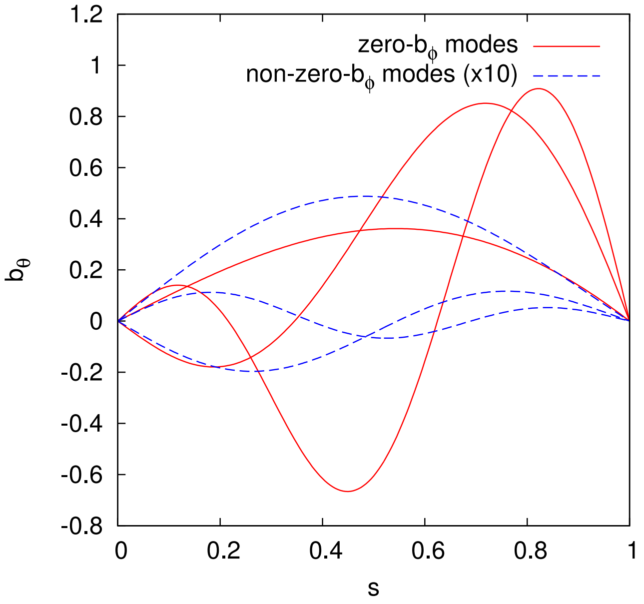
$$\gamma \boldsymbol{b} = -\nabla \times \bar{\boldsymbol{e}} \qquad \langle \bar{\boldsymbol{w}}, \boldsymbol{b} \rangle = \langle \bar{\boldsymbol{w}}, \bar{\boldsymbol{b}} \rangle$$

$$\boldsymbol{e} = \eta \nabla \times \bar{\boldsymbol{b}} \qquad \langle \boldsymbol{w}, \bar{\boldsymbol{e}} \rangle = \langle \boldsymbol{w}, \boldsymbol{e} \rangle$$

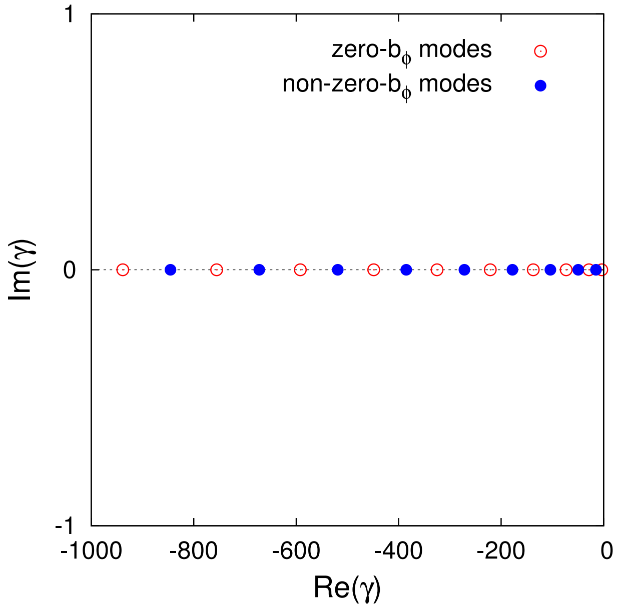
Boundary condition (s=1) : $b_\theta = b_\phi = 0$

Simulation parameters :

$$m = n = 1, \ R_0 = 2, \ \eta = 1.$$



Eigenmodes



Eigenvalues

Eigenmode	Simulation (N=200)	Analysis
1st zero-b_φ	-3.63400	-3.63994
1st non-zero-b_φ	-14.9322	-14.9320
2nd zero-b_φ	-28.6759	-28.6743
2nd non-zero-b_φ	-49.4719	-49.4685
3rd zero-b_φ	-73.1290	-73.1188
3rd non-zero-b_φ	-103.766	-103.750

- Guarantee negative real eigenvalues
- No spurious eigenmode
- Excellent agreement with the analytic solution

Application to MHD initial value problem

Weak formulation of
the fully implicit (Backward differentiation (BDF) algorithm)
linear resistive MHD equations

$$\left\langle \boldsymbol{w}_v, -\nabla p^n \right\rangle + \left\langle \boldsymbol{w}_v, \boldsymbol{j}_{(0)} \times \boldsymbol{b}^n \right\rangle + \left\langle \boldsymbol{w}_v, (\nabla\times\boldsymbol{b}^n) \times \boldsymbol{b}_{(0)} \right\rangle + \left\langle \boldsymbol{w}_v, \nu\, \nabla\times\nabla\times\bar{\boldsymbol{v}}^n \right\rangle$$

$$+\left\langle \boldsymbol{w}_v, -\frac{3}{2\Delta t}\rho_0 \boldsymbol{v}^n \right\rangle = \left\langle \boldsymbol{w}_v, \rho_0 \left(-\frac{2}{\Delta t}\boldsymbol{v}^{n-1} + \frac{1}{2\Delta t}\boldsymbol{v}^{n-2} \right) \right\rangle$$

$$\left\langle w_p, -\boldsymbol{v}^n \cdot \nabla p_{(0)} \right\rangle + \left\langle w_p, -\Gamma p_{(0)} \nabla \cdot \boldsymbol{v}^n \right\rangle + \left\langle w_p, -\frac{3}{2\Delta t} p^n \right\rangle$$

$$= \left\langle w_p, -\frac{2}{\Delta t} p^{n-1} + \frac{1}{2\Delta t} p^{n-2} \right\rangle$$

$$\left\langle \bar{\boldsymbol{w}}_b, -\nabla\times\bar{\boldsymbol{e}}^n \right\rangle + \left\langle \bar{\boldsymbol{w}}_b, -\frac{3}{2\Delta t}\boldsymbol{b}^n \right\rangle = \left\langle \bar{\boldsymbol{w}}_b, -\frac{2}{\Delta t}\boldsymbol{b}^{n-1} + \frac{1}{2\Delta t}\boldsymbol{b}^{n-2} \right\rangle$$

$$\left\langle \bar{\boldsymbol{w}}_e, \bar{\boldsymbol{e}}^n \right\rangle = \left\langle \bar{\boldsymbol{w}}_e, -\boldsymbol{v}^n \times \boldsymbol{b}_0 \right\rangle + \left\langle \bar{\boldsymbol{w}}_e, \eta\nabla\times\bar{\boldsymbol{b}}^n \right\rangle$$

Verification and Validation of MHD Initial Value Code (1)

m/n=2/1 Suydam mode

Cylindrical tokamak : $R_0/a = 5$

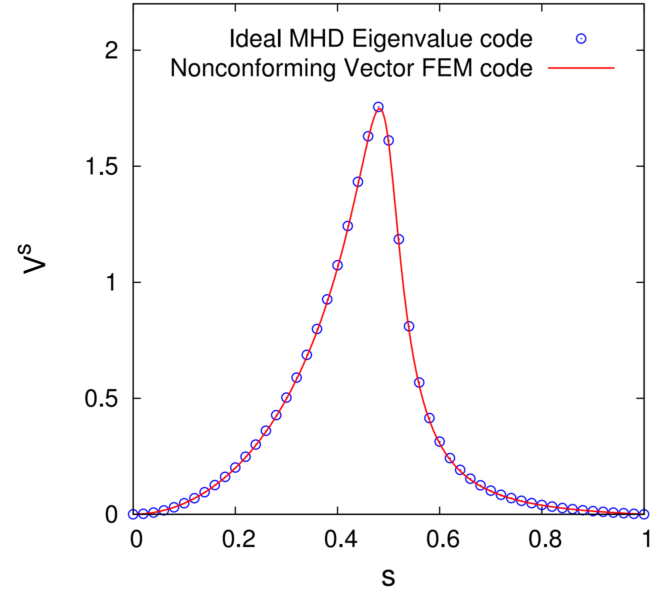
Resonant surface (q=2) position : $s_{mn} = 0.5$

Suydam index : D = 0.588

Radial grid points : N = 500

Time step : $\Delta t/\tau_A = 1$

Ideal and Resistive : $\eta = 0, 10^{-6}$



Eigenmode

Growth rate

$1/N^2$ convergence : The best resolution expected

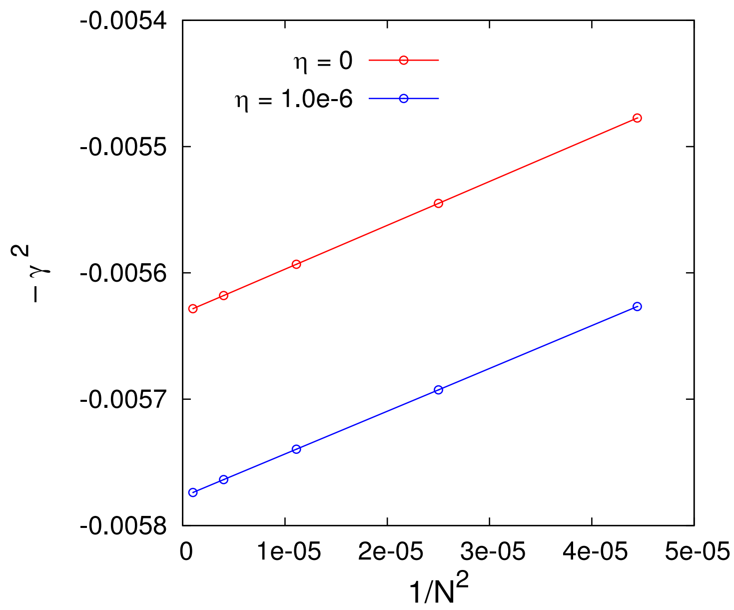
Nonconforming VFEM initial-value code

Ideal MHD case : -5.632×10^{-3}

Resistive ($\eta=10^{-6}$) case : -5.777×10^{-3}

Ideal MHD eigenvalue code[1] : -5.858×10^{-3}

[1] R. Gruber and J. Rappaz: Finite Element Methods in Linear Ideal Magnetohydrodynamics (Springer-Verlag, Berlin, 1985).



Grid convergence

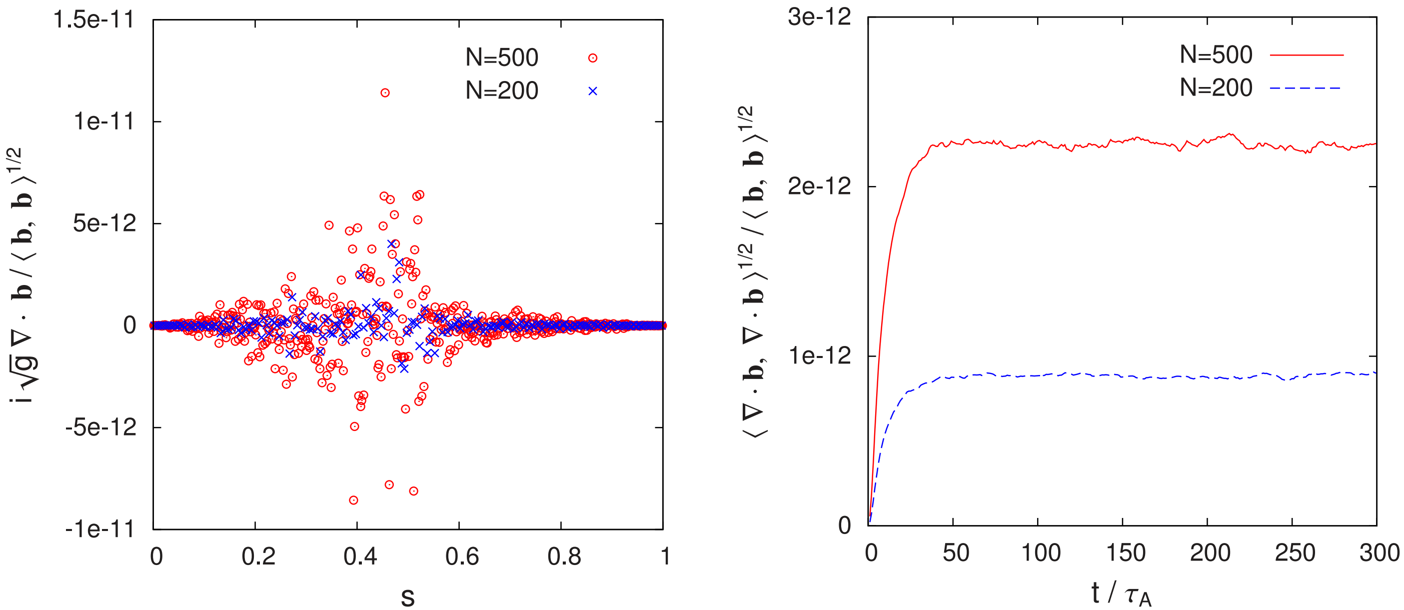
Verification of the Divergence Constraint

Magnetic field and velocity field divergence

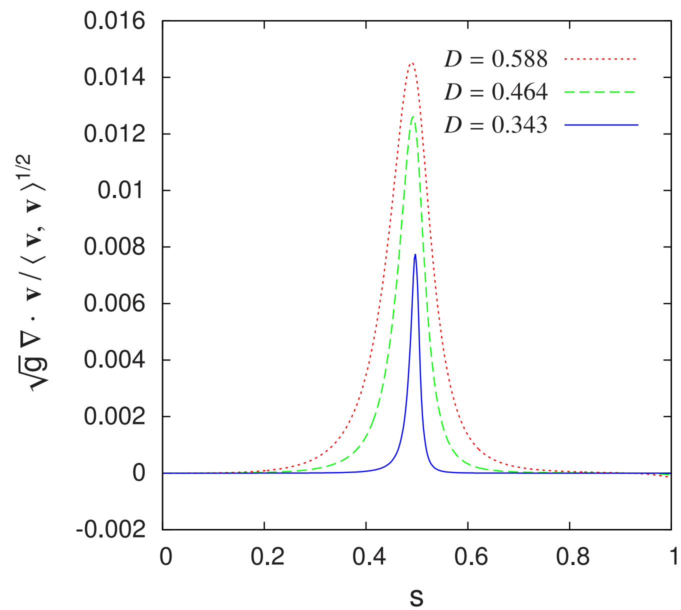
m/n=2/1 Suydam mode

Radial mesh number : N = 200, 500

Suydam index :
D = 0.343, 0.464, 0.588



$\nabla\cdot\boldsymbol{b}$, Radial profile and temporal variation



$\nabla\cdot\boldsymbol{v}$, Radial profile

$\nabla\cdot\boldsymbol{b}$: Error of only about 10^{-12} due to the discretization of the spatial derivatives.

$\nabla\cdot\boldsymbol{v}$: The divergence violation around the resonant surface tends toward zero as Suydam index parameter (D) is varied toward 1/4, i.e., Suydam criterion.

Verification and Validation of MHD Initial Value Code (2)

m/n=2/1 Resistive internal kink mode

Cylindrical tokamak ： $R_0/a = 5$

Resonant surface (q=2) position ： $s_{mn} = 0.5$

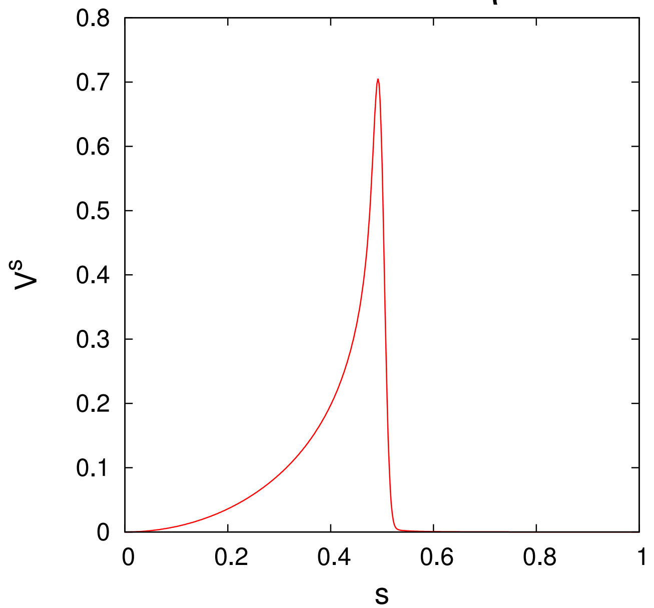
Suydam index ： D = 0.221　Stable against Suydam mode

Radial grid points ： N = 500

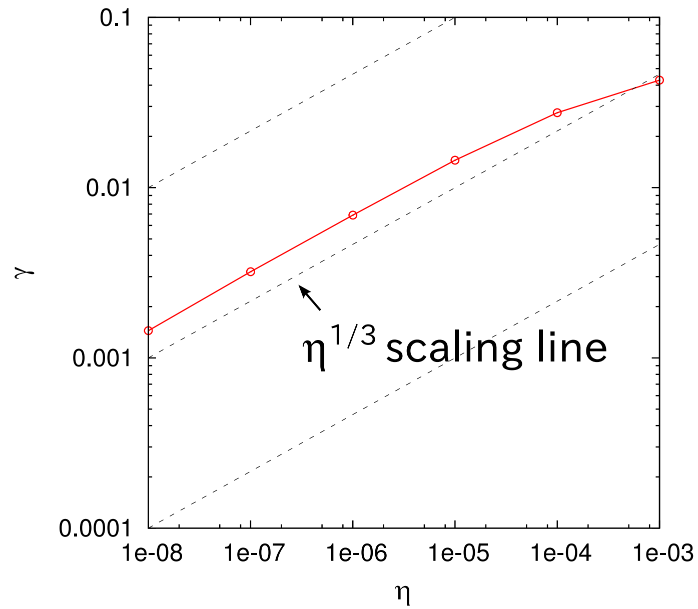
Time step ： $\Delta t/\tau_A = 1$

The growth rate scales as $\eta^{1/3}$, that assures the code works as expected.

$\eta = 10^{-6}$



Eigenmode



Resistivity convergence

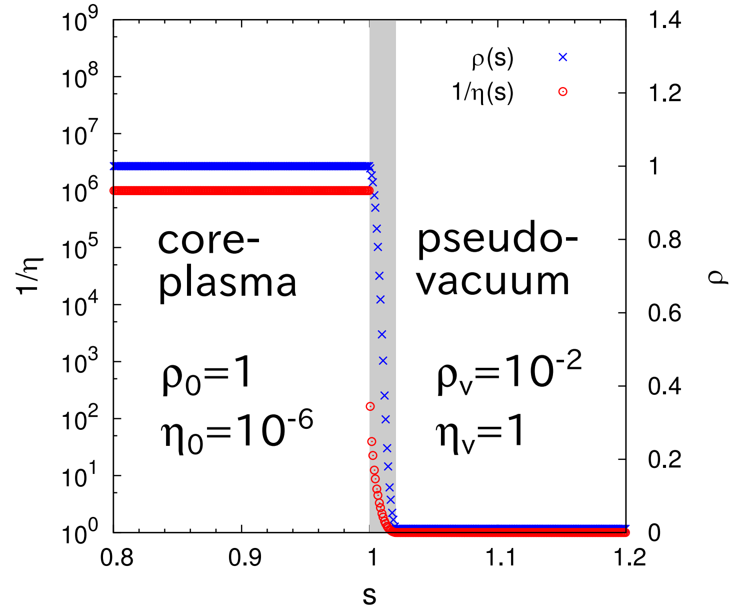
Verification and Validation of MHD Initial Value Code (3)

Free boundary simulation by using a pseudo-vacuum model

$0 < s < 1$: Core-plasma

$1 < s < 1+\delta$: Transition interlayer

$1+\delta < s < b$: Pseudo-vacuum
(Highly resistive, low density plasma)



Resistivity and mass density profile

m/n=2/1 external kink mode

Cylindrical tokamak : $R_0/a = 5$

Boundary wall : $b/a = 1.25$

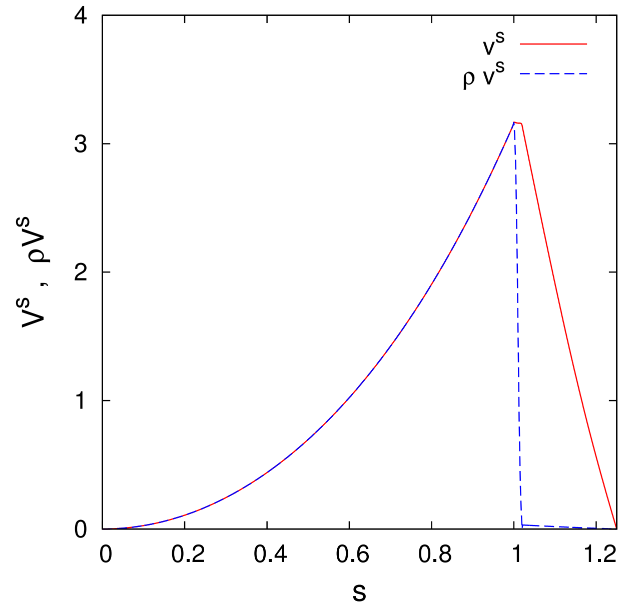
Safety factor : $q_0 = 1.37$, $q_a = 1.65$

Resistivity : $\eta_0 = 10^{-6}$, $\eta_v = 1$

Mass density : $\rho_0 = 1$, $\rho_v = 10^{-2}$

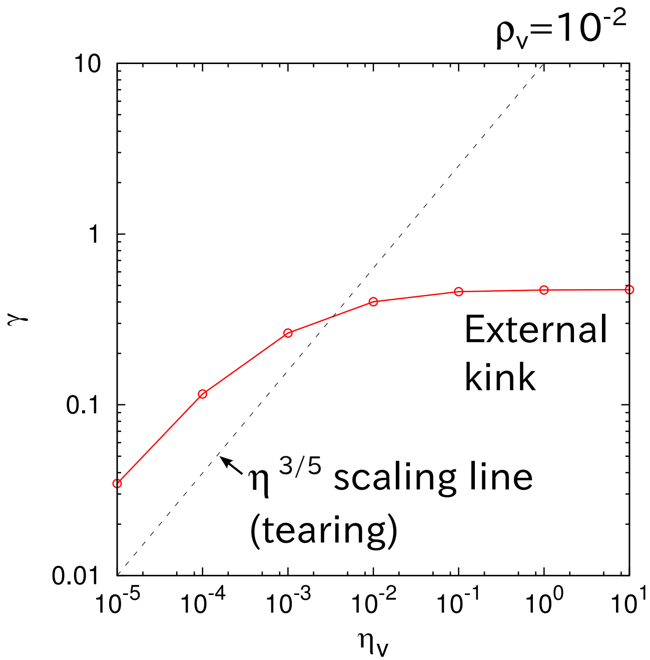
Radial grid points : $N_a = 1000$, $N_v = 250$

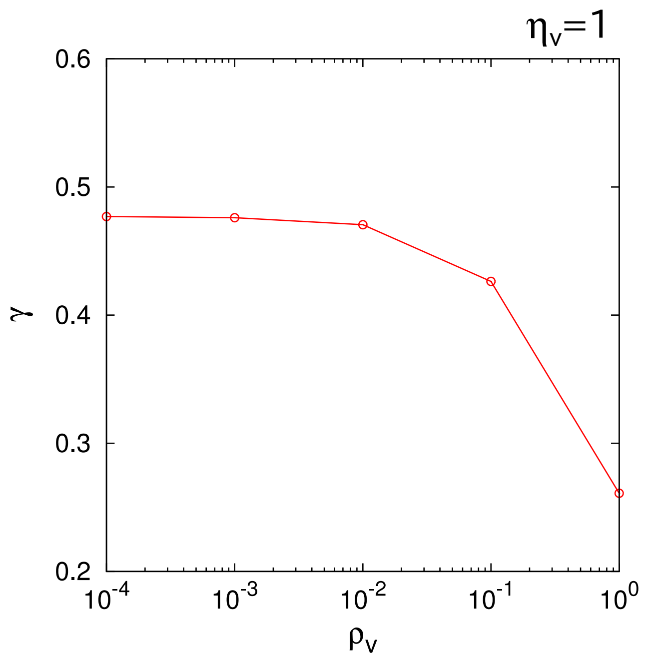
Time step : $\Delta t/\tau_A = 0.1$

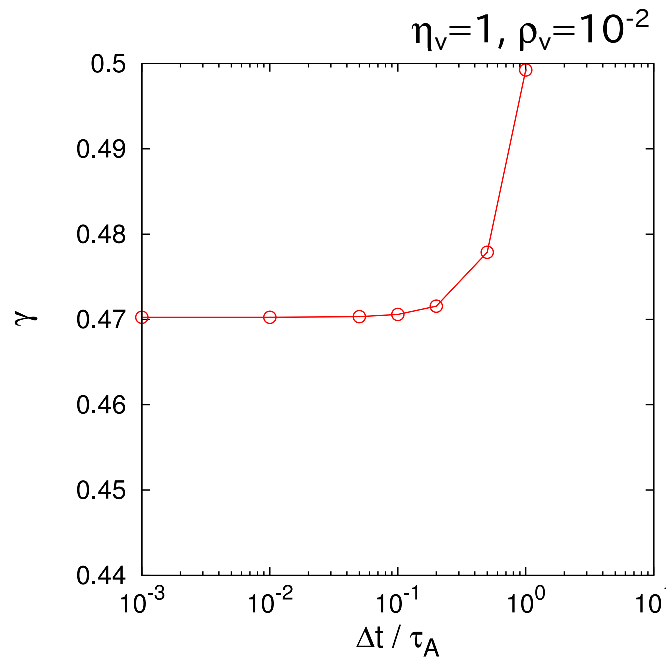


Eigenmode

Verification and Validation of MHD Initial Value Code (4)







Vacuum resistivity (η_V) and vacuum mass density (ρ_V) dependences of the growth rate

Time steps dependence of the growth rate

$\eta_v/\eta_0 > 10^{-1}/10^{-6}$ and $\rho_v/\rho_0 < 10^{-2}$ are required. $\Rightarrow$ A severe restriction is imposed on the time step size if an explict approach is used.

Fully implicit method allows time steps of $0.1\tau_A$ (τ_A : poloidal Alfven time).

100 times larger order of the spatial grid size, Δs

1000 times larger order of $\left(\sqrt{\rho_v/\rho_0}\right)\Delta s$

Summary

A novel vector finite element method is proposed.

The method consists of two factors ;

1. Basis functions of covariant and contravaiant vectors are determined individually according to the applicability of them to the discrete 'curl' operator and the discrete 'divergence' operator.

2. Covariant-contravariant metric transformation is given by the weak form in which the norm conserving condition is imposed.

This kind of method, called 'Nonconforming vector finite element method', is implemented in a single-fluid resistive MHD code.

Numerical experiments demonstrate excellent performances,
in particular, the divergence-free condition is confirmed to be satisfied.