Particle flux representations with FLR effects in the gyrokinetic model

ジャイロ運動論モデルにおける有限ラーマー半径効果を含む粒子フラックスの表現

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Introduction

• In general, simple fluid moments of the gyro-center distribution function (N, P, ...) are different from physical fluid moments (n, p,...).
• In particular, gyro-center particle flux in standard gyrokinetic model does not include diamagnetic and polarization-drift terms.
• We have to consider push-forward representation of particle flux to recover the diamagnetic and polarization-drift terms.
Standard gyrokinetic formulation

[Hahm PoF 1988, Brizard JPP 1989]

Two-step phase space transformation to remove fast gyro-motion of a charged particle in a magnetic field.

\[ Z = (x, v) \]

\[ Z = (X, U, \mu, \xi) \]

Small potential perturbation \( \varphi \) is introduced after \( T_{GC} \).

\( T_{GC} \): guiding-center transformation (small parameter \( \epsilon_B \))

\( T_{Gy} \): gyro-center transformation (small parameter \( \epsilon_\delta \))
Particle flux representation

Two abstract push-forward representations of particle flux \( \Gamma \equiv n[\mathbf{v}] \)

**Conventional**
\[
\Gamma(r) = \int d^6Z J[T_{GC}^{-1}v](Z)[T_{Gy}^*F](Z)\delta^3([T_{GC}^{-1}x](Z) - r)
\]

**Pure push-forward**
\[
\Gamma(r) = \int d^6Z J[T_{Gy}^{-1}T_{GC}^{-1}vF](Z)\delta^3(T_{Gy}^{-1}T_{GC}^{-1}x - r)
\]

- Conventionally the first representation is used to derive explicit push-forward representations of fluid moments from the standard gyrokinetic model.
- They give the same result, but more information on the gyro-center transformation is needed to derive the polarization flux from the conventional representation as shown below.
Pull-back transformation of $F$

Generally

$$T_{Gy}^* F = F + \varepsilon \delta \{ S_1, F \} + \frac{\varepsilon^2}{2} \{ S_1, \{ S_1, F \} \} + \varepsilon \delta \{ S_2, F \} + O(\varepsilon^3)$$

$S_1, S_2, ...$ : Scalar function generating the gyro-center transformation.

$\{ , \} :$ Guiding-center Poisson brackets [Littlejohn PoF 1981].

Equation determining $S_1$ [Brizard JPP 1989, Qin PoP 1998]

$$\frac{\partial S_1}{\partial t} + \{ S_1, H_0 \} = e \tilde{\phi}$$

where

Guiding-center Hamiltonian

$$H_0 = \frac{m}{2} U^2 + \mu B$$

Oscillatory part of potential

$$\tilde{\phi} = \phi(\mathbf{X} + \mathbf{\rho}_0) - \langle \phi(\mathbf{X} + \mathbf{\rho}_0) \rangle$$

Gyro-average

$\uparrow$
Generating function $S_1$

Usually only the lowest order solution for $S_1$ is considered:

$$S_1^{(1)} = \frac{e}{\Omega} \int \tilde{\phi} \, d\xi$$

Then $T^*_G F$ is approximated as

$$T^*_G F \approx F + \frac{e\tilde{\phi}}{B} \frac{\partial F}{\partial \mu}$$

But we cannot obtain the polarization flux from the conventional representation with the lowest order $S_1$, because time derivative of potential does not appear at this order.

To obtain the polarization flux, we need the higher order solution for $S_1$ [Qin PoP 1999, Belova 2001]

$$S_1^{(2)} = -\frac{1}{\Omega} \left( \frac{\partial}{\partial t} + U b \cdot \nabla \right) \int d\xi S_1^{(1)}$$

where a constant magnetic field is assumed for simplicity.
Gyro-center displacement vector

When we use the pure push-forward representation, the higher order solution is not necessary.

In the pure push-forward representation

\[ T_{\text{Gy}}^{-1} T_{\text{GC}}^{-1} \mathbf{x} = \mathbf{X} + \rho_0 + \rho_1 + \cdots \]
\[ T_{\text{Gy}}^{-1} T_{\text{GC}}^{-1} \mathbf{v} = \dot{\mathbf{x}} + \dot{\rho}_0 + \dot{\rho}_1 + \cdots \]

Gyro-center displacement vector [Brizard PoP 2008]

\[ \rho_1 \equiv -\{S_1, \mathbf{X} + \rho_0\} \]
\[ = \frac{e}{m} \left( \frac{\partial S_1}{\partial \mu} \frac{\partial \rho_0}{\partial \xi} - \frac{\partial S_1}{\partial \xi} \frac{\partial \rho_0}{\partial \mu} \right) + \frac{\mathbf{B}^*}{mB^*_{\parallel}} \frac{\partial S_1}{\partial \mu} + \frac{1}{eB^*_{\parallel}} \mathbf{b} \times \nabla S_1 \]

In long wavelength limit, gyro-average of \( \rho_1 \) with the lowest order \( S_1 \) is given by

\[ \langle \rho_1 \rangle = -\frac{\nabla \times \varphi}{B \Omega} \]

The polarization flux emerges from \( \dot{\rho}_1 \) directly.
Belova derived an explicit representation of $\Gamma$ with FLR terms from the conventional representation with $S_1^{(2)}$ and $S_2$ as well as $S_1^{(1)}$ [Belova PoP 2001].

From the above observations, we may obtain Belova’s result from the pure push-forward representation using only $S_1^{(1)}$.

We calculate $\Gamma$ using the pure push-forward representation with $S_1^{(1)}$.

Generally, $\Gamma$ is divided into 3 parts [Pfirsch ZNA 1984, Brizard PoP 2008]:

\[ \Gamma = \Gamma_{gy} + \Gamma_{pol} + \Gamma_{mag} \]

\[ \Gamma_{gy} = \int dU d\mu d\xi \dot{X} F J : \text{gyro-center flux} \]

\[ \Gamma_{pol} = \frac{\partial}{\partial t} P: \text{polarization flux (P is polarization vector)} \]

\[ \Gamma_{mag} = \nabla \times M: \text{magnetization flux (M is magnetization vector)} \]
Gyro-center flux

We consider a slab plasma, then Hamilton equations are

\[
\dot{X} = U b + \frac{1}{B} b \times \nabla \Phi, \quad \dot{U} = -e b \cdot \nabla \Phi, \quad \dot{\mu} = 0, \quad \dot{\xi} = \Omega + \frac{e \Omega}{B} \frac{\partial \Phi}{\partial \mu}
\]

Effective potential is given by

\[
\Phi = \langle \varphi(X + \rho_0) \rangle - \frac{1}{2} \langle \{S_1, \tilde{\varphi} \} \rangle
\]

Using the lowest order solution for $S_1$ and taking the long wavelength limit, we have

\[
\Phi(X, \mu, t) \approx \varphi(X, t) + \frac{\mu}{2e \Omega} \nabla^2 \varphi(X, t) - \frac{1}{2B \Omega} |\nabla \varphi(X, t)|^2
\]

(This is the same one in a gyro-kinetic model for flowing plasmas [Miyato JPSJ 2009])

Gyro-center flux is obtained immediately

\[
\Gamma_{gy} = NV_{\parallel} b + Nv_E - \frac{N}{2B \Omega} b \times \nabla |\nabla \varphi|^2 + \frac{P_{\perp}}{2m \Omega^2} \frac{b \times \nabla \varphi}{B}
\]
Polarization flux \cdot Magnetization flux

Polarization flux

\[
\Gamma_{\text{pol}} = \frac{\partial}{\partial t} \int d^3v \left[ \langle \rho_1 \rangle FJ - \frac{1}{2} \nabla \cdot (\langle \rho_0 \rho_0 \rangle FJ) \right] = -\frac{\partial}{\partial t} \left[ \frac{N}{B\Omega} \nabla_\perp \varphi + \frac{1}{2m\Omega^2} \nabla_\perp P_\perp \right]
\]

Magnetization flux

\[
\Gamma_{\text{mag}} = \nabla \times \left[ -\frac{P_\perp}{eB} b - \frac{P_\perp^2 \nabla_\perp^2 \varphi}{m\Omega^2 B} b - \frac{3}{2} \frac{\nabla_\perp P_\perp \cdot \nabla_\perp \varphi}{m\Omega^2 B} b - \nabla_\perp^2 \frac{M_{20}}{4e^2\Omega} b - N\frac{\nabla_\perp \varphi^2}{B^2\Omega} b + N\frac{V_\parallel}{\Omega} v_E - \nabla \times \frac{M_{11} B}{2m\Omega^2} b \right]
\]

\[
\approx \left[ \nabla \cdot \left( \frac{NV_\parallel}{B\Omega} \nabla_\perp \varphi \right) + \nabla_\perp^2 \frac{M_{11}}{2e\Omega} \right] b
\]

This modifies \( NV_\parallel b \) in \( \Gamma_{\text{gy}} \)
Total particle flux

\[ \Gamma = \left[ NV_{\parallel} + \nabla_{\perp}^{2} \frac{M_{11}}{2e\Omega} + \nabla \cdot \left( \frac{NV_{\parallel}}{B\Omega} \nabla_{\perp} \varphi \right) \right] b \]

\[ + Nv_{E} - \frac{N}{2B\Omega} \frac{b \times \nabla |\nabla_{\perp} \varphi|^{2}}{B} + \frac{P_{\perp}}{2m\Omega^{2}} \frac{b \times \nabla \nabla_{\perp}^{2} \varphi}{B} \]

\[ - \frac{\partial}{\partial t} \left[ \frac{N}{B\Omega} \nabla_{\perp} \varphi + \frac{1}{2m\Omega^{2}} \nabla_{\perp} P_{\perp} \right] \]

\[ + \nabla \times \left[ - \frac{P_{\perp}}{eB} b - \frac{P_{\perp} \nabla_{\perp}^{2} \varphi}{m\Omega^{2} B} b - \frac{3}{2} \frac{\nabla_{\perp} P_{\perp} \cdot \nabla_{\perp} \varphi}{m\Omega^{2} B} b - \frac{\nabla_{\perp} M_{20}}{4e^{2}\Omega} b - N \frac{\nabla_{\perp} \varphi|^{2}}{B^{2}\Omega} b \right] \]

Using a vector relation

\[ pb \times \nabla \nabla_{\perp}^{2} \varphi - (\nabla_{\perp}^{2} p) b \times \nabla \varphi \]

\[ = b \times \nabla (p \nabla_{\perp}^{2} \varphi) - b \times \nabla \{ (\nabla_{\perp} \varphi) \cdot (\nabla_{\perp} p) \} \]

\[ - 2b \times \nabla \cdot \nabla \nabla_{\perp} \varphi - \nabla_{\perp} (b \times \nabla \varphi \cdot \nabla p) \]

to delete the 3rd term in the 2nd line,
Recover Belova’s result

\[
\Gamma = \left[ NV_{\parallel} + \nabla_{\perp}^2 \frac{M_{11}}{2e\Omega} + \nabla \cdot \left( \frac{NV_{\parallel}}{B\Omega} \nabla_{\perp} \varphi \right) \right] b
\]

\[+Nv_E - \frac{N}{2B\Omega} b \times \nabla |\nabla_{\perp} \varphi|^2 + \frac{\nabla_{\perp}^2 P_{\perp}}{2m\Omega^2} v_E\]

\[-\frac{\partial}{\partial t} \frac{N}{B\Omega} \nabla_{\perp} \varphi - \frac{b \times \nabla P_{\perp} \cdot \nabla_{\perp} \varphi}{m\Omega^2 B}\]

\[-\frac{1}{2m\Omega^2} \nabla_{\perp} \left[ \frac{\partial P_{\perp}}{\partial t} + v_E \cdot \nabla P_{\perp} \right]\]

\[+\nabla \times \left[ -\frac{P_{\perp}}{eb} b - \frac{P_{\perp} \nabla_{\perp}^2 \varphi}{m\Omega^2 B} b - \frac{3}{2} \frac{\nabla_{\perp}^2 P_{\perp} \cdot \nabla_{\perp} \varphi}{m\Omega^2 B} b - \nabla_{\perp}^2 \frac{M_{20}}{4e^2\Omega} b - N \frac{|\nabla_{\perp} \varphi|^2}{B^2\Omega} b \right]\]

Neglecting the 4\textsuperscript{th} line recovers Belova’s result obtained from the conventional representation.

This is small
Alternate representation

Using another vector relation,

\[
pb \times \nabla \nabla_\perp^2 \varphi - (\nabla_\perp^2 p) b \times \nabla \varphi \\
= \ b \times \nabla(p \nabla_\perp^2 \varphi) + \nabla_\perp (b \times \nabla \varphi \cdot \nabla p) \\
- 2b \times \nabla \varphi \cdot \nabla \nabla_\perp p - b \times \nabla \{ (\nabla_\perp \varphi) \cdot (\nabla_\perp p) \}
\]

and \( \mathbf{v}_E \cdot \nabla P_\perp \simeq -\partial_t P_\perp \)

\[
\Gamma = \left[ NV_\parallel + \nabla_\perp^2 \frac{M_{11}}{2 e \Omega} + \nabla \cdot \left( \frac{NV_\parallel}{B \Omega} \nabla_\perp \varphi \right) \right] b \\
+ N \mathbf{v}_E - \frac{N}{2 B \Omega} \frac{b \times \nabla |\nabla_\perp \varphi|^2}{B} + \frac{\nabla_\perp^2 P_\perp}{2 m \Omega^2} \mathbf{v}_E \\
- \frac{\partial}{\partial t} \frac{N}{B \Omega} \nabla_\perp \varphi - \frac{1}{m \Omega^2} \left( \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \nabla_\perp P_\perp \\
+ \nabla \times \left[ -\frac{P_\perp}{e B} \ b - \frac{3 P_\perp}{2} \frac{\nabla_\perp^2 \varphi}{m \Omega^2 B} \ b - \frac{\nabla_\perp P_\perp \cdot \nabla_\perp \phi}{m \Omega^2 B} \ b - \nabla_\perp^2 \frac{M_{20}}{4 e^2 \Omega} \ b - N \frac{|\nabla_\perp \varphi|^2}{B^2 \Omega} b \right]
\]
Transformation to particle fluid moments

Using push-forward representations of particle fluid moments

\[
\begin{align*}
n &= N + \frac{1}{2eB\Omega} \nabla_\perp^2 P_\perp + \nabla \cdot \left(\frac{N}{\Omega B} \nabla_\perp \varphi\right) \\
n u_\parallel &= N V_\parallel + \nabla_\perp^2 \frac{M_{11}}{2e\Omega} + \nabla \cdot \left(\frac{N V_\parallel}{\Omega B} \nabla_\perp \varphi\right) \\
p_\perp &= P_\perp + \frac{B}{2e\Omega} \nabla_\perp^2 M_{20} + 2\nabla \cdot \left(\frac{P_\perp}{\Omega B} \nabla_\perp \varphi\right)
\end{align*}
\]

Gyrofluid moments are transformed as

\[
\Gamma = n u_\parallel \mathbf{b} + n \mathbf{v}_E \\
- \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right) \frac{n}{B\Omega} \nabla_\perp \varphi - \frac{1}{m\Omega^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right) \nabla_\perp p_\perp \\
+ \nabla \times \left[ -\frac{p_\perp}{eB} \mathbf{b} + \frac{1}{2} \frac{p_\perp \nabla^2 \varphi}{m\Omega^2 B} \mathbf{b} + \frac{\nabla_\perp p_\perp \cdot \nabla_\perp \varphi}{m\Omega^2 B} \mathbf{b} + \nabla^2 \frac{m_{20}}{4e^2 \Omega} \mathbf{b} \right]
\]

A FLR corrected particle continuity equation is obtained immediately.
\[
\partial_t n + \nabla \cdot \Gamma = 0
\]
Summary

• We have derived an explicit push-forward representations of particle flux with FLR terms including the polarization drift term from the pure push-forward abstract representation using only the lowest order solution of $S_1, S_1^{(1)}$, in the standard electrostatic gyrokinetic model.

• Belova’s result, which was derived from the conventional representation with $S_1^{(2)}, S_2$ as well as $S_1^{(1)}$, has been recovered from our result through a vector relation and neglecting small terms.

• We have also derived the other form of representation and then the FLR corrected continuity equation retaining the time derivative term of $\nabla_\perp p_\perp$ by transforming the gyrofluid moments to the particle fluid moments.

• Electromagnetic generalization in the $p_z$ formulation [Hahm-Lee-Brizard 1988] is easier for the pure push-forward representation.