Development of a robust scheme for compressible MHD

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# MHD code projects

For laboratory plasmas

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Any information or corrections are appreciated…

- Extended MHD model
- Not designed for shock capturing
Shocks in space plasmas

Coronal activities (Hinode)

Ubiquitous reconnection / jet (Hinode)

Magnetosphere (SCOPE)

Shock wave

Magnetic Reconnection

Boundary Layer Turbulence

Processes of fundamental importance in the Plasma Universe

Magnetosphere (Artist's image/NASA)

Heliosphere (Artist's image/NASA)

Astrospheres

Jets from Young Stars

HST - WFPC2

Jet from black hole (ATCA)
Motivation and objectives

- In compressible MHD codes for laboratory plasmas, time integration methods have been polished so as to solve stiff problems.

- But, those codes are not designed for shock capturing that may be needed in the near future. (e.g., HiFi code at PSI-Center)

- Shocks and turbulence are universally observed in space. Thus, the development of robust shock capturing schemes has been highly progressed.

- Current status and challenges of the shock capturing scheme for MHD are presented with emphasis on our results.
Compressible MHD equations

- Ideal MHD equations (Non-conservative form)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad : \text{continuity equation} \]

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \quad : \text{equation of motion} \]

\[ \frac{\partial}{\partial t} \left( \frac{p}{\rho^\gamma} \right) + \mathbf{v} \cdot \nabla \left( \frac{p}{\rho^\gamma} \right) = 0 \quad : \text{adiabatic equation} \]

\[ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \quad \nabla \cdot \mathbf{B} = 0 \quad : \text{induction equation} \]

- Various non-conservative forms can be obtained using vector identities
Compressible MHD equations

- Ideal MHD equations *(Conservative form)*

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad : \text{mass conservation} \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + p_T \mathbf{I} - \mathbf{B} \mathbf{B}) &= 0 \quad : \text{momentum conservation} \\
\frac{\partial e}{\partial t} + \nabla \cdot \left[ (e + p_T) \mathbf{v} - \mathbf{B} (\mathbf{v} \cdot \mathbf{B}) \right] &= 0 \quad : \text{energy conservation} \\
\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) &= 0 \quad : \text{flux conservation} \\
\nabla \cdot \mathbf{B} &= 0 \ , \quad p = (\gamma - 1) \left( e - \frac{\rho v^2}{2} - \frac{B^2}{2} \right) \ , \quad p_T = p + \frac{B^2}{2}
\end{align*}
\]
Shock capturing scheme

- Non-conservative scheme
  - Based on non-conservative form
  - Converge to unphysical shock
    Hou-LeFloch [1994]

- Conservative scheme
  - Based on conservative form
  - Converge to physical shock
    Lax-Wendroff [1960]
    Harten [1980]
  - Difficult to preserve positivity

Conservative vs Non-conservative

“Computational Tutorial: MHD”, Toth
Shock capturing scheme

- Non-conservative scheme
  - Finite difference method
  - Finite element method

- Conservative scheme
  - Finite difference method
    - FD-WENO, Compact FD+LAD, etc.
  - Finite element method
    - RKDG, etc.
  - Finite volume method
    - MUSCL, FV-WENO, etc.
Shock capturing scheme

1D finite volume method

\[ \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \Rightarrow \frac{d}{dt} \bar{u}_i + \frac{f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t))}{\Delta x} = 0 \]

\[
\begin{array}{c|c|c|c}
\text{f}_{i-1/2} & \bar{u}_{i-1} & \text{f}_{i+1/2} & \bar{u}_{i+2} \\
\text{x}_{i-1/2} & \text{ } & \text{x}_{i+1/2} & \text{ }
\end{array}
\]

Numerical flux

\[ f(u(x_{i+1/2}, t)) \equiv f_{i+1/2} = f(\cdots, \bar{u}_{i-1}, \bar{u}_i, \bar{u}_{i+1}, \bar{u}_{i+2}, \cdots) \]
Approximate Riemann solver

- Approximate Riemann solver

\[ \int \int \left( \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \right) dx dt = \int \left( Ud\mathbf{x} - F dt \right) = 0 \]

- Define piecewise constants

\[ U \]

\[ x \]
Approximate Riemann solver

- Approximate Riemann solver

\[ \int\int \left( \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \right) dx dt = \int (U dx - F dt) = 0 \]

- Define piecewise constants
- Solve local Riemann problems
Riemann problem

- Riemann problem = Shock tube problem
- 7-waves can be excited in 1D MHD system (shock, expansion wave, compound wave)

\[ U = U\left(\frac{x}{t}; U_R, U_L\right) \]

\[ FS / FR \quad RD \quad SS / SR \quad CD \quad SS / SR \quad RD \quad FS / FR \]

\[ U_L \quad U_R \]
Approximate Riemann solver

- Approximate Riemann solver

\[
\iint \left( \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \right) dx \, dt = \oint (U \, dx - F \, dt) = 0
\]

- Define piecewise constants
- Solve local Riemann problems
- Average state variables
Approximate Riemann solver

- Define piecewise constants
- Solve local Riemann problems
- Average state variables
- Derive numerical fluxes from conservation laws

\[ \iint \left( \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \right) dx dt = \int (U dx - F dt) = 0 \]

\[ \int_{x_i}^{x_{i+1/2}} U \left( \frac{x - x_{i+1/2}}{\Delta t}; U_i^n, U_{i+1}^n \right) dx - (x_{i+1/2} - x_i) U_i^n + \Delta t (F_{i+1/2} - F_i^n) = 0 \]

Depend on “quality” of approximate solutions!
Approximate Riemann solver

- Standard approximate Riemann solver
  - Lax-Friedrichs scheme [Lax, 1950’s]
  - Godunov scheme [Godunov, 1959]
  - Rusanov scheme [Rusanov, 1961]
  - Roe scheme (HD) [Roe, 1981]
  - HLL scheme [Harten+ 1983]
  - Roe scheme (MHD) [Brio+, 1988]
  - HLLC scheme (HD) [Toro+, 1994; Batten+ 1997]
  - HLLC scheme (MHD) [Gurski, 2004; Li, 2005]
  - HLLD scheme (MHD) [Miyoshi+, 2005]
HLL approximate Riemann solver

- HLL Riemann solver [Harten+, 1983]
  - Conservation laws
  - 2-waves approximation

\[ \begin{align*}
  &S_L, S_R: \text{max./min. speeds} \\
  &S_R = \max(u_L + c_L, u_R + c_R, 0) \\
  &S_L = \min(u_L - c_L, u_R - c_R, 0) \\
  &\int (Udx - Fdt) = 0 \implies (S_R - S_L)U^* - S_R U_R + S_L U_L + F_R - F_L = 0 \\
  &\text{CD/TD/RD cannot be resolved}
\end{align*} \]
HLL approximate Riemann solver

- HLL Riemann solver [Harten+, 1983]
  - Conservation laws
  - 2-waves approximation

\[
S_{R,L} : \max/\min \text{ speeds}
\]

\[
S_R = \max(u_L + c_L, u_R + c_R, 0)
\]

\[
S_L = \min(u_L - c_L, u_R - c_R, 0)
\]

\[
\int (Udx - Fdt) = 0 \Rightarrow S_RU^* - S_RU_R + F_R - F^* = 0
\]

- CD/TD/RD cannot be resolved
HLLD approximate Riemann solver

- HLLD Riemann solver [Miyoshi+, 2005]
  - Conservation laws
  - 5-waves approximation

\[
S_{R,L} \left( U_{R,L}^* - U_{R,L} \right) = F_{R,L}^* - F_{R,L}, \quad S_{R,L}^* \left( U_{R,L}^* - U_{R,L}^* \right) = F_{R,L}^{**} - F_{R,L}^*,
\]

\[
S_M \left( U_R^{**} - U_L^{**} \right) = F_R^{**} - F_L^{**}, \quad \frac{1}{\Delta t} \int_{S_L \Delta t} S_R U \left( x, t^{n+1} \right) dx + S_R U_R - S_L U_L + F_R - F_L = 0
\]

\[S_{R,L}^* : \text{fast magnetosonic wave}\]
\[S_M^* : \text{entropy wave}\]
\[S_{R,L}^* : \text{Alfvén wave}\]
HLLD approximate Riemann solver

- The HLLD Riemann solver
  - is constructed without eigenvectors
  - exactly resolves isolated CD/TD/RD/FS
  - preserves density and pressure positivities

- High-efficiency! High-resolution! Robust!
HLLD approximate Riemann solver

- Established as a standard Riemann solver
  - Comparing numerical methods [Kritsuk+, 2011]

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Notes.
- See Section 3 and the indicated sections on each topic for more information.
- Base method. FD for finite difference, FV for finite volume. FV techniques have the Riemann solver listed, Section 6.3.
- Spatial order of accuracy, Section 6.1.
- Artificial Viscosity, Section 6.2. “$\pi$ Derivative” indicates presence of terms proportional to the longitudinal derivative of the magnetic field.
- MHD method, Section 6.4.
- Time integration method, Section 6.6.3.
- Multidimensional technique, Section 6.6.2. “$\perp$ Reconstruction” indicates presence of transverse derivatives in the interface reconstruction.

- Athena (US), CANS+ (Japan), and many other researches
Challenges

- Challenges to multi-D MHD scheme

  Comparing numerical methods [Kritsuk+, 2011]

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Challenges to multi-D

- Treatment of numerical magnetic monopole
  - Negative effect due to unphysical magnetic force
    \[-\nabla \cdot \left( B^2 / 2 I - BB \right) = (\nabla \times B) \times B + B(\nabla \cdot B)\]
  - Need divergence-free/divergence-cleaning method!
Challenges to multi-D

Treatment of numerical magnetic monopole

Can numerical simulations preserve \( \nabla \cdot \mathbf{B} = 0 \)?
Challenges to multi-D

- Numerical shock instabilities
  - Odd-even decoupling
  - Carbuncle phenomena
Challenges

- Challenges to higher-order MHD scheme
  - Comparing numerical methods [Kritsuk+, 2011]

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Challenges to higher-order

- Importance of higher-order methods
  - Error of $n$th-order method vs. Computational cost

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \]
\[ u(x, t = 0) = \sin(2\pi x) \]
\[ L1 \equiv \frac{1}{N} \sum_{i} |u_i - u(x_i, t)| \]
Challenges to higher-order

- Godunov’s theorem
  - Any linear monotone scheme (non-oscillatory scheme) can be at most first-order accurate.
  - This statement suggests that higher-order non-oscillatory scheme can be constructed as a nonlinear scheme.
  - TVD, MUSCL, PPM, WENO, etc.

Very-high-order WENO (up to 17th-order) [Gerolymos+, 2009]
Challenges

- *Multi-dimensional higher-order divergence-free* scheme is one of the goals of shock capturing scheme for MHD
Summary

- I have reported current status and challenges of robust shock capturing schemes for MHD
  - The HLLD has been established as a standard MHD solver in the field of astrophysics
  - Multi-D shock capturing scheme for MHD is one of the challenges
    - Treatment of numerical magnetic monopole
    - Treatment of numerical shock instabilities
  - Higher-order shock capturing scheme for MHD is one of the challenges
  - Study on shock capturing scheme for two-fluid / extended MHD is now progressing…