Linear Analysis of Energetic-Particle-Driven Low-Frequency Eigenmodes

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Outline

- Full wave approach of linear stability analysis
- Full wave code: TASK/WM
- Analysis of Alfvén eigenmodes
  - TAE
  - EPM
  - TAE in rotating plasmas
  - RSAE
- Progress in full wave analysis
- Summary
Conventional analysis of global stability analysis

- **MHD Analysis** (Ideal, Resistive)
- **Extended MHD Analysis** (Hall, Multi-fluid)
- **MHD including Kinetic Effect** (perturbative)
  - Eigen function from MHD analysis
  - Growth rate including kinetic effects

Issues in MHD analysis

- Propagation in vacuum
- Strongly coupled with plasma model

Full wave approach to global stability analysis

- Boundary value problem of Maxwell’s equation
- Dielectric tensor describes plasma response
- Physical damping mechanism
Damping Mechanism of Alfvén Eigenmodes

• MHD model
  – Absorption near Alfvén resonance
    (Continuous spectrum damping)

• Perturbative treatment of kinetic Alfvén waves
  (Eigen function: MHD, Damping: Kinetic)
  – Radiative damping
    (power propagating outward)
  – Landau damping
    (Estimation of parallel wave electric field)

• Kinetic absorption mechanism
  – Electron Landau damping
  – Landau damping of energetic ions
Integrated code for the analysis of toroidal plasmas

- EQ: 2D fixed-/free-boundary equilibrium
- EX: 2D anisotropic pressure equilibrium
- TR: 1D diffusive transport
- TX: 1D dynamic transport
- T2: 2D dynamic transport
- FP: 1D kinetic transport (3D Fokker-Planck)
- WR: 3D ray/beam tracing
- WM: 3D full wave analysis (2D FFT+ 1D FEM)
- WF: 3D full wave analysis (1D FFT+ 2D FEM)
- DP: Wave dispersion relation
- FIT3D: NBI analysis (birth, orbit, deposit)
- EG: Gyrokinetic linear microinstability
- GNET: Drift-Kinetic equation solver
- KITES: 3D MHD equilibrium
Full wave code in TASK

- **Features of full wave component**: TASK/WM
  - Boundary value problem of Maxwell’s equation
  - Various models of dielectric tensor: TASK/DP
  - Magnetic surface coordinates from MHD Equilibrium Analysis
  - Fourier mode expansion in poloidal and toroidal direction
  - Finite difference method in radial direction
  - Complex wave frequency to maximize the wave field.

- **Other full wave components**
  - Using finite element method

- **Coupling with Fokker-Planck analysis of** $f(v)$: TASK/FP
  - Generation of energetic particles
• Magnetic surface coordinates: \((\psi, \theta, \varphi)\)
  – Non-orthogonal system (including 3D helical configuration)

• Maxwell’s equation for stationary wave electric field \(E\)
  
  \[ \nabla \times \nabla \times E = \frac{\omega^2}{c^2} \mathbf{\epsilon} \cdot E + i \omega \mu_0 j_{\text{ext}} \]
  – \(\mathbf{\epsilon}\) : Dielectric tensor with kinetic effects: \(Z[(\omega - n\omega_c)/k_{||}]\)

• Fourier expansion in poloidal and toroidal directions
  – Exact parallel wave number: \(k_{||}^{m,n} = (mB^\theta + nB^\varphi)/B\)

• Destabilization by energetic ions included in \(\mathbf{\epsilon}\)
  – Drift kinetic equation
  
  \[ \left[ \frac{\partial}{\partial t} + v_{||} \nabla_{||} + (v_d + v_E) \cdot \nabla + \frac{e\alpha}{m\alpha} (v_{||}E_{||} + v_d \cdot E) \frac{\partial}{\partial \epsilon} \right] f_\alpha = 0 \]

• Eigenvalue problem for complex wave frequency
  – Maximize wave amplitude for finite excitation proportional to \(n_e\)
Coordinates

- **Magnetic Surface Coordinates** (Non-Orthogonal)
  - Minor radius direction: Poloidal Magnetic Flux $\psi$
  - Poloidal direction: $\theta$
  - Toroidal direction: $\varphi$

- **Co-variant expression** of $E$

  $$E = E_1 e^1 + E_2 e^2 + E_3 e^3$$

  where contra-variant basis

  $$e^1 = \nabla \psi, \quad e^2 = \nabla \theta, \quad e^3 = \nabla \varphi$$

- **$J$: Jacobian**

  \[
  J = \frac{1}{e^1 \cdot e^2 \times e^3} = \frac{1}{\nabla \psi \cdot \nabla \theta \times \nabla \varphi}
  \]

- **$g$: Metric tensor**

  $$g_{ij} = e_i \cdot e_j,$$ where co-variant basis $e_i \equiv \partial r / \partial x_i$
Wave Equation

- **Maxwell’s equation** for stationary wave electric field $E$ (angular frequency $\omega$, light velocity $c$)

$$\nabla \times \nabla \times E = \frac{\omega^2}{c^2} \varepsilon \cdot E + i \omega \mu_0 j_{\text{ext}}$$

- $\varepsilon$: Dielectric tensor: plasma response
  - Cyclotron damping, Landau damping
- $j_{\text{ext}}$: Antenna Current

- **Wave equation in non-orthogonal coordinates** (radial components)

$$(\nabla \times \nabla \times E)^1 = \frac{1}{J} \left[ \frac{\partial}{\partial x^2} \left\{ g_{31} \left( \frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) + g_{32} \left( \frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) + g_{33} \left( \frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \right\} \right.$$ 

$$- \frac{\partial}{\partial x^3} \left\{ g_{21} \left( \frac{\partial E_3}{\partial x^3} - \frac{\partial E_2}{\partial x^2} \right) + g_{22} \left( \frac{\partial E_1}{\partial x^2} - \frac{\partial E_3}{\partial x^3} \right) + g_{23} \left( \frac{\partial E_2}{\partial x^3} - \frac{\partial E_1}{\partial x^2} \right) \right\}$$

- $(x^1, x^2, x^3) = (\psi, \theta, \varphi)$
- Similar expression for poloidal and toroidal components
Response of Plasmas

- Usually the dielectric tensor $\varepsilon$ is calculated in Cartesian coordinates with static magnetic field along the $z$ axis.

- **Local normalized orthogonal coordinates**
  \[ \hat{e}_s = \frac{\nabla \psi}{|\nabla \psi|}, \quad \hat{e}_b = \hat{e}_h \times \hat{e}_\psi, \quad \hat{e}_h = \frac{B_0}{|B_0|} \]

- **Variable Transformation:** $\mu$
  \[
  \begin{pmatrix}
  E_1 \\
  E_2 \\
  E_3 
  \end{pmatrix}
  = \mu \cdot
  \begin{pmatrix}
  E_s \\
  E_b \\
  E_h 
  \end{pmatrix}
  \]

  \[
  \mu = \begin{pmatrix}
  \frac{1}{\sqrt{g^{11}}} & \frac{d}{\sqrt{J g^{11}}} & c_2 g_{12} + c_3 g_{13} \\
  0 & c_3 J \sqrt{g^{11}} & c_2 g_{22} + c_3 g_{23} \\
  0 & -c_2 J \sqrt{g^{11}} & c_2 g_{32} + c_3 g_{33} 
  \end{pmatrix}
  \]

  \[
  c_2 = B^0 / B, \quad c_2 = B^\phi / B \\
  d = c_2 (g_{23} g_{12} - g_{22} g_{31}) + c_3 (g_{33} g_{12} - g_{32} g_{31}) \\
  g^{11} = (g_{22} g_{33} - g_{23} g_{32}) / J^2 
  \]

- **Dielectric tensor in non-orthogonal coordinates:**
  \[ \varepsilon = \mu \cdot \varepsilon_{sbh} \cdot \mu^{-1} \]
**Fourier Mode Expansion**

- **Fourier expansion in poloidal and toroidal directions**

- **Spatial variation of wave electric field, medium and the L.H.S. of Maxwell’s equation**

\[
E(\psi, \theta, \varphi) = \sum_{mn} E_{mn}(\psi) e^{i(m\theta + n\varphi)}
\]

\[
G(\psi, \theta, \varphi) = \sum_{lk} G_{lk}(\psi) e^{i(l\theta + kN_p\varphi)}
\]

\[
J(\nabla \times \nabla \times E) = G(\psi, \theta, \varphi) E(\psi, \theta, \varphi) = \sum_{m'n'} [J(\nabla \times \nabla \times E)]_{m'n'} e^{i(m'\theta + n'\varphi)}
\]

- **Coupling between various modes** \((N_h : \text{Rotation number of helical coil in } \varphi)\)

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Toroidal Direction</th>
<th>Poloidal Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave electric field (E)</td>
<td>(n)</td>
<td>(m)</td>
</tr>
<tr>
<td>Medium (G)</td>
<td>(kN_h)</td>
<td>(l)</td>
</tr>
<tr>
<td>(J(\nabla \times \nabla \times E))</td>
<td>(n')</td>
<td>(m')</td>
</tr>
<tr>
<td>Relations</td>
<td>(n' = n + kN_h)</td>
<td>(m' = m + l)</td>
</tr>
</tbody>
</table>
Parallel Wave Number

- **Dielectric tensor** \( \vec{\varepsilon} (\psi, \theta, \varphi, k''''_{n''}) \) depends on parallel wave number \( k''''_{n''} \) through the **plasma dispersion function** \( Z[(\omega - N\omega_{cs})/k''''_{n''} v_{Ts}] \)

\[
k''''_{n''} = -i\hat{e}_h \cdot \nabla = -i\hat{e}_h \cdot (\nabla \theta \frac{\partial}{\partial \theta} + \nabla \varphi \frac{\partial}{\partial \varphi})
\]

\[
= -i\hat{e}_h \cdot (e^2 \frac{\partial}{\partial \theta} + e^3 \frac{\partial}{\partial \varphi}) = m'' \frac{B^\theta}{|B|} + n'' \frac{B^\varphi}{|B|}
\]

- **Fourier components of electric displacement**

\[
(J \leftrightarrow \vec{E})^i = J \leftrightarrow \mu^{-1} \leftrightarrow \mu \leftrightarrow \varepsilon_{sbh} \leftrightarrow \mu^{-1} \cdot E_i
\]

\[
m' \quad \ell_3 \quad \ell_2 \quad \ell_1 \quad m
\]

\[
n' \quad k_3 \quad k_2 \quad k_1 \quad n
\]

therefore

\[
m'' = m + \ell_1 + \frac{1}{2} \ell_2 \quad n'' = n + k_1 + \frac{1}{2} k_2
\]

\[
m' = m + \ell_1 + \ell_2 + \ell_2 \quad n' = n + k_1 + k_2 + k_3
\]
Destabilization by Energetic Ion

- **Drift kinetic equation**

\[
\left[ \frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + (v_d + v_E) \cdot \nabla + \frac{e_\alpha}{m_\alpha} (v_\parallel E_\parallel + v_d \cdot E) \frac{\partial}{\partial \varepsilon} \right] f_\alpha = 0
\]

where

\[
\varepsilon = \frac{1}{2} m_\alpha v^2, \quad v_E = \frac{E \times B}{B^2}, \quad v_d = v_d \sin \theta \hat{r} + v_d \cos \theta \hat{\theta},
\]

\[
v_d = \frac{m_\alpha}{e_\alpha BR} \cdot \frac{v_\perp^2}{v_\perp^2 + v_\parallel^2}
\]

- **Linear response**
  - Velocity integral of perturbed distribution function
  - Poloidal mode coupling due to magnetic drift motion
Response of Energetic Particles

- **Anti-Hermite part of electric susceptibility tensor**

\[
\vec{\chi}_{mm'} = \begin{pmatrix}
1 & -i & 0 \\
-i & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} P_{m-1,m-2}\delta_{m',m-2} + \begin{pmatrix}
0 & 0 & Q_{m-1,m-1} \\
0 & 0 & -i Q_{m-1,m-1} \\
Q_{m,m-1} & -i Q_{m,m-1} & 0
\end{pmatrix} \delta_{m',m-1}
\]

\[
+ \begin{pmatrix}
(P_{m-1,m} + P_{m+1,m}) & i(P_{m-1,m} - P_{m+1,m}) & 0 \\
-i(P_{m-1,m} - P_{m+1,m}) & (P_{m-1,m} + P_{m+1,m}) & 0 \\
0 & 0 & R_{m-1,m-1}
\end{pmatrix} \delta_{m',m} + \begin{pmatrix}
0 & 0 & Q_{m+1,m+1} \\
0 & 0 & i Q_{m+1,m+1} \\
Q_{m,m+1} & i Q_{m,m+1} & 0
\end{pmatrix} \delta_{m',m+1} + \begin{pmatrix}
1 & i & 0 \\
i & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} P_{m+1,m+2}\delta_{m',m+2}
\]

- **In the case of Maxwellian velocity distribution**

\[
P_{m,m'} = i \frac{\omega_{p\alpha}^2}{2\omega^2} \left(1 - \frac{\omega_{*m'}}{\omega}\right) \frac{\rho_{\alpha}^2}{R^2} \sqrt{\pi} x_m \left(\frac{1}{2} + x_m^2 + x_m^4\right) e^{-x_m^2}
\]

\[
Q_{m,m'} = i \frac{\omega_{p\alpha}^2}{2\omega^2} \left(1 - \frac{\omega_{*m'}}{\omega}\right) \frac{\rho_{\alpha}}{R} \sqrt{\pi} 2x_m^2 \left(\frac{1}{2} + x_m^2\right) e^{-x_m^2}
\]

\[
R_{m,m'} = i \frac{\omega_{p\alpha}^2}{2\omega^2} \left(1 - \frac{\omega_{*m'}}{\omega}\right) \sqrt{\pi} 4x_m^3 e^{-x_m^2}
\]

\[
x_m = \omega / |k||m|v_{T\alpha}, \quad \rho_{\alpha} = v_{T\alpha} / \omega_{\alpha}, \quad v_{T\alpha} = \sqrt{2T_{\alpha}/m_{\alpha}}
\]
Boundary Conditions

- Calculation region surrounded by **perfectly conducting wall**
  (*Vacuum region* exists between plasma surface and wall)

- Boundary condition on the **conducting wall**:
  Tangential components of $E$ vanishes.
  - Co-variant expression ($E = E_1 \nabla \psi + E_2 \nabla \theta + E_3 \nabla \varphi$): $E_2 = 0, \ E_3 = 0$

- Boundary condition on the **magnetic axis** ($\psi = 0$):
  Finiteness of the wave magnetic field and the induced charge density

\[
\begin{align*}
  m = 0 & \quad \frac{\partial E_\varphi^{0n}}{\partial \psi} = 0 \\
  m \neq 0 & \quad E_\varphi^{mn} = 0
\end{align*}
\]

Co-variant component $E_\theta^{mn}$ always vanishes on the axis.
Typical TAE with Positive Magnetic Shear

- Configuration
  - \( q(\rho) = q_0 + (q_a - q_0)\rho^2 \), \( q_0 = 1 \), \( q_a = 2 \)
  - Flat Density Profile

Contour of \(|E|^2\) in Complex Frequency Space

Alfvén Frequency

Eigen function

\( f_r = 81.95 \text{ kHz} \)
\( f_i = -20.32 \text{ Hz} \)
Energetic Particle Mode (EPM)

- Energetic ions can excite EPM with frequency below the TAE frequency gap.
- With $\beta$ of energetic ions about 0.5%, $\omega_A$ and contour of wave amplitude

- Eigenmode structure

\[ A: f_r = 41.8 \text{ kHz} \]
\[ A: f_i = 8.1 \text{ kHz} \]
\[ B: f_r = 57.7 \text{ kHz} \]
\[ B: f_i = 3.6 \text{ kHz} \]
\[ C: f_r = 58.0 \text{ kHz} \]
\[ C: f_i = -6.0 \text{ kHz} \]
Parameter Dependence of Mode Structure

\[ n_{F0} = 0 \times 10^{17} \text{ m}^{-3}, T_B = 0.5 \text{ MeV} \]

\[ n_{F0} = 1 \times 10^{17} \text{ m}^{-3}, T_B = 0.5 \text{ MeV} \]

\[ n_{F0} = 3 \times 10^{17} \text{ m}^{-3}, T_B = 0.5 \text{ MeV} \]

\[ n_{F0} = 1 \times 10^{17} \text{ m}^{-3}, T_B = 1 \text{ MeV} \]
Effect of Toroidal Plasma Rotation

Experimental Results on JT-60U

Co-NBI $\rightarrow$ Counter-NBI

Counter-NBI $\rightarrow$ Co-NBI

Counter-NBI: stabilization

Co-NBI: destabilization
Dispersion Relation including Toroidal Rotation

- **Dispersion relation**

\[
\left(k_{||m}^2 - \frac{(\omega - k_{||m}u)^2}{v_A^2}\right) \left(k_{||m+1}^2 - \frac{(\omega - k_{||m+1}u)^2}{v_A^2}\right) - \epsilon^2 \frac{(\omega - k_{||m}u)^2(\omega - k_{||m+1}u)^2}{v_A^4} = 0
\]

- **Parallel wave number** \( @k_{||m} = \frac{1}{R} \left( n + \frac{m}{q} \right) \)

- **Alfvén resonance condition without toroidal effect**

\[
\omega^2 = k_{||m}^2 (u \pm v_A)^2, \quad \omega^2 = k_{||m+1}^2 (u \pm v_A)^2
\]

- **Condition for frequency gap**

\[
k_{||m} (u - v_A) = k_{||m+1} (u + v_A)
\]

- **Safety factor at TAE gap**: \( q \)

\[
q = -\frac{m + 1/2}{n} - \frac{1}{2n v_A} u
\]

- **TAE gap frequency** \( \omega \): parabolic with respect to \( u \)

\[
\omega = \frac{v_A}{2qR} \left( 1 - \frac{u^2}{v_A^2} \right)
\]
Effect of Rotation on $n = 1$ mode

$n = 1$ Eigenmode for JT-60U parameters

Dependence of eigen frequency and damping rate on Rotation Velocity and Velocity Gradient

$$U_t = U_r(1 - \rho^2)^\alpha \left/ \left(1 - \rho_r^2\right)^\alpha \right.$$
Effect of Rotation on $n = 7$ mode

- $n = 7, m = -17 \sim -3, f = 223 \text{ kHz}$  Good agreement with Nova-K

Rotation velocity dependence: Stabilizing for co rotation (Contradict with exp.)
Influence of poloidal mode range: \( n = 7 \) mode

- Radial structure of Alfvén Continuum

\[ m = -17 \sim -3 \quad m = -21 \sim -7 \]

- \( n = 7, m = -21 \sim -7, f = 238 \text{ kHz} \) (Destabilizing for co-rotation (agree with exp.))
AE in the Reversed Magnetic Shear Configuration (JT-60U)

- Takechi et al. IAEA 2002 (Lyon) EX/W-6

**Fluctuation Amplitude**

**Observed frequency** vs **calculated frequency**

![Graph showing fluctuation amplitude and frequency analysis](image-url)
First observation of RSAE by TASK/WM


IAEA Technical Committee Meeting on
Alpha-Particles in Fusion Research
September 8-11, 1997
JET, Abingdon, UK

Kinetic Analysis of TAE in Tokamaks and Helical Devices

A. Fukuyama and T. Tohmai
Faculty of Engineering, Okayama University, Okayama

Fig 4: Radial profile of $q$ (a), resonance frequency (b) and eigen function (c) in the case of negative shear: $q(0) = 3$, $q_{\text{min}} = 2$, $q(a) = 5$ and $n = 1$.

Fig 5: $q_{\text{min}}$ dependence of the eigen frequency; real part (a) and imaginary part (b)
Analysis of AE in Reversed Shear Configuration

**Assumed q profile**

Plasma Parameters

\[
\begin{align*}
R_0 &= 3 \text{ m} \\
a &= 1 \text{ m} \\
B_0 &= 3 \text{ T} \\
n_e(0) &= 10^{20} \text{ m}^{-3} \\
T(0) &= 3 \text{ keV} \\
q(0) &= 3 \\
q(a) &= 5 \\
\rho_{\text{min}} &= 0.5 \\
n &= 1 \\
\end{align*}
\]

Flat density profile

- **RSAE** (reversed-shear-induced Alfvén eigenmode) for \( \ell + \frac{1}{2} < q_{\text{min}} < \ell + 1 \)
$q_{\text{min}}$ Dependence of Radial Structure of Alfvén resonance

\[ q_{\text{min}} = 2.0 \quad q_{\text{min}} = 2.1 \quad q_{\text{min}} = 2.2 \quad q_{\text{min}} = 2.3 \]

\[ q_{\text{min}} = 2.4 \quad q_{\text{min}} = 2.5 \quad q_{\text{min}} = 2.6 \quad q_{\text{min}} = 2.7 \]

\[ q_{\text{min}} = 2.8 \quad q_{\text{min}} = 2.9 \quad q_{\text{min}} = 3.0 \]
Eigenmode Structure \((n = 1)\)

- \(q_{\text{min}} = 2.4\)
- \(q_{\text{min}} = 2.5\)
- \(q_{\text{min}} = 2.6\)

**Alfvén resonance**

**Higher freq.**

**Lower freq.**

TAEs  Double TAE  RSAE
Excitation by Energetic Particles ($q_{\text{min}} = 2.6$)

- **Without EP**
  - $E_{\theta}$
  - $n_F = 0 \text{ m}^{-3}$

- **With EP**
  - $n_F = 3 \times 10^{16} \text{ m}^{-3}$
    - $E_{\theta}$
    - $f_r = 38.0 \text{ kHz}$
    - $f_i = 160.2 \text{ Hz}$
    - $m = 3$
    - $E_{\theta}$
    - $f_r = 55.3 \text{ kHz}$
    - $f_i = 1858.6 \text{ Hz}$
    - $m = 3$

- **With EP**
  - $n_F = 1 \times 10^{17} \text{ m}^{-3}$
    - $E_{\theta}$
    - $f_r = 37.2 \text{ kHz}$
    - $f_i = 271.6 \text{ Hz}$
    - $m = 3$
Progress in Full Wave Analysis

• Variety of numerical schemes

<table>
<thead>
<tr>
<th>module</th>
<th>system</th>
<th>scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>WM</td>
<td>torus</td>
<td>toroidal &amp; poloidal: FFT, radial: FDM</td>
</tr>
<tr>
<td>WMF</td>
<td>torus</td>
<td>toroidal &amp; poloidal: FFT, radial: FEM</td>
</tr>
<tr>
<td>WF2D</td>
<td>torus</td>
<td>toroidal: FFT, poloidal and radial: FEM</td>
</tr>
<tr>
<td>WF3D</td>
<td>Cartesian</td>
<td>$x, y, z$: FEM</td>
</tr>
</tbody>
</table>

  – Merit of FEM: Flexibility of mesh, sparse matrix, localized analysis

• Extension of dielectric tensor

  – Uniform, kinetic, Maxwellian, Fourier expansion
  – Nonuniform, gyro kinetic, Maxwellian, Fourier expansion
  – Nonuniform, kinetic, Maxwellian, Integral form
  – Uniform, kinetic, arbitrary $f(v)$, Fourier expansion
  – Nonuniform, gyro kinetic, arbitrary $f(v)$, Fourier expansion

• Coupling with Fokker-Planck analysis of $f(v)$
Momentum Distribution Functions

- Radial diffusion proportional to $E^{-1/2}$ reduces energetic ions in the outer region.
Summary

- **Full wave approach** of linear stability analysis is powerful for systematic analysis of various kinds of global eigenmodes.
  - Alfvén eigenmodes
  - Resistive wall mode, internal kink mode, · · ·

- **Kinetic effects** of energetic particles and bulk species can be included in the dielectric tensor, though non-uniformity and gyro-kinetic effects may complicate the derivation.

- A variety of **Alfvén eigenmodes** have been analyzed by TASK/WM and the results were compared with other codes and experimental observations.

- Large scale computer will enable us to carry out **systematic parameter survey** in more realistic plasma models for future reactors.