Analytical and Numerical Studies on Acceleration Phase of Collisionless Magnetic Reconnection

M. Hirota\textsuperscript{1}

in collaboration with P. J. Morrison\textsuperscript{2}, Y. Ishii\textsuperscript{1}, M. Yagi\textsuperscript{1}, N. Aiba\textsuperscript{1}

\textsuperscript{1}Japan Atomic Energy Agency

\textsuperscript{2}University of Texas at Austin

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Introduction

- Magnetic reconnection is triggered by dissipation/microscopic effects.
  (singular perturbations of ideal MHD)

- If plasma is either collisionless ($R_m \sim 10^{9-13}$) or close to the ideal MHD stability limit ($\Delta' \sim \infty$), the resistive MHD theory cannot explain the observed reconnection speeds. ⇒ collisionless magnetic reconnection

Numerical simulations show acceleration of collisionless reconnection in nonlinear phase


- However, conventional methods (such as asymptotic matching and perturbation expansion) have difficulty in analysing the nonlinear evolution.

- We take a new theoretical approach based on variational principle in order to clarify the acceleration mechanism. Our analytical prediction is also verified by using a direct numerical simulation.
Triggers of reconnection in two-fluid model

Faraday’s law
$\partial_t B = -\nabla \times E$

$E = -\mathbf{v} \times \mathbf{B} + \frac{d_i}{n} (\mathbf{j} \times \mathbf{B} - \nabla p_e) + \frac{d_e^2}{n} \frac{d\mathbf{j}}{dt} + \eta \mathbf{j} - \eta_2 \nabla^2 \mathbf{j}$

$\mathbf{v}$: ion velocity, $n$: number density, $\mathbf{j}$: current, $p_e$: electron pressure

(1). Hall effect: $d_i = \text{(ion skin depth)}/L$
$\partial_t B = \nabla \times (v_e \times \mathbf{B})$ where $v_e = \mathbf{v} - d_i j/n$ \quad \cdots \text{no reconnection, by itself}

(2). Electron inertia: $d_e = \text{(electron skin depth)}/L$ \quad \cdots \text{collisionless reconnection}

(3). Resistivity: $\eta$ \quad \cdots \text{collisional reconnection}

Ref. Rutherford theory (linear phase $\propto e^{\gamma t}$ $\Rightarrow$ nonlinear phase $\propto t$)

(4). Electron viscosity: $\eta_2$ \quad \cdots \text{collisional reconnection}

In large tokamaks, \quad \boxed{1 \gg 2 \gtrapprox 3 \gg 4}

We will focus on electron inertia (2) and study nonlinear acceleration mechanism of collisionless reconnection.
Analytical model of this work

2D MHD model with electron inertia

For \( \mathbf{v} = \nabla \phi(x, y, t) \times \mathbf{e}_z \) and \( \mathbf{B} = \nabla \psi(x, y, t) \times \mathbf{e}_z \),

Vorticity equation:

\[
\frac{\partial \nabla^2 \phi}{\partial t} - [\phi, \nabla^2 \phi] - [\nabla^2 \psi, \psi] = 0,
\]

(1)

(Collisionless) Ohm’s law:

\[
\frac{\partial (\psi - d_e^2 \nabla^2 \psi)}{\partial t} - [\phi, \psi - d_e^2 \nabla^2 \psi] = 0,
\]

(2)

where \( d_e (\ll L) \): electron skin depth, and \([f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}\).

This is known as a Hamiltonian system. (no dissipation)

- Hamiltonian: \( H = \frac{1}{2} \int d^2 x \left[ |\nabla \phi|^2 + |\nabla \psi|^2 + d_e^2 (\nabla^2 \psi)^2 \right] \)

- Ohm’s law (2) \( \Leftrightarrow \partial_t \psi_e + \mathbf{v} \cdot \nabla \psi_e = 0 \)

Instead of magnetic flux \( \psi \), electron’s canonical momentum \( \psi_e = \psi - d_e^2 \nabla^2 \psi \) is the frozen-in flux.

\( \Rightarrow \) Reconnection is possible without any dissipation mechanism.
We consider

1D equilibrium (periodic in both $x$ and $y$ directions)

$$\phi \equiv 0 \text{ (no flow)}, \quad \psi(x) = \cos \frac{2\pi x}{L_x} \quad \text{on } \left[-\frac{L_x}{2}, \frac{L_x}{2}\right]$$

- Collisionless magnetic reconnection spontaneously occurs at resonant surfaces $x = 0, \pm L_x/2$.
- For sufficiently small wavenumber $k$ in the $y$ direction, this instability ($\Delta' \sim \infty$) is similar to the $m = 1$ kink-tearing mode in tokamaks.

The reconnection process mainly leads to the following energy conversion:

$$\frac{1}{2} \int |\nabla \psi|^2 d^2x \quad \text{Relaxation} \quad \frac{1}{2} \int |\nabla \phi|^2 d^2x \quad \text{and} \quad \frac{1}{2} \int d_e^2 J^2 d^2x \quad (J = -\nabla^2 \psi)$$
Direct numerical simulation

[Finite difference method in $x$ direction ($\sim 10,000$ grids), Spectral method in $y$ direction ($\sim 100$ modes)]

Define $\epsilon$ as maximum displacement in $x$ direction ($\approx$ half width of magnetic island).

Snapshots of contours when $\epsilon = 4.2d_e$ ($d_e/L_x = 0.01$, $k = 0.5/L_x$)

\[ \frac{\epsilon}{d_e} \text{ indeed accelerates when it exceeds the electron skin depth } d_e. \]

[Ottaviani and Porceli, PRL (1993)]
Construction of variational principle

♦ Perturbations, \((0, \psi_e) \rightarrow (\tilde{\phi}, \tilde{\psi}_e)\), that preserve the flux \(\psi_e\) can be generated by a function \(G(x, y, t)\) such that

\[
\begin{align*}
\tilde{\phi}(x + \partial_y G(x, y, t), y, t) &= \partial_t G(x, y, t), \\
\tilde{\psi}_e(x + \partial_y G(x, y, t), y, t) &= \psi_e(x)
\end{align*}
\]

\[\Rightarrow \text{Ohm's law } (2) \text{ is solved! (which is built-in as a constraint on } \tilde{\phi} \text{ and } \tilde{\psi}_e)\]

♦ Lagrangian: \(L[G] = \frac{1}{2} \int \left( |\nabla \tilde{\phi}|^2 - |\nabla \tilde{\psi}|^2 - d_e^2 |\nabla^2 \tilde{\psi}|^2 \right) d^2x = K - W\)

This play a role of potential energy

Variational principle: \(\delta \int L[G]dt = 0 \text{ w.r.t. } \forall \delta G \Rightarrow \text{Vorticity eq. } (1)\)

If the potential energy decreases \((\delta W < 0)\) for some function \(G\), then such a perturbation will grow with the release of free energy.

(The MHD energy principle is extended to two-fluid model.)
Linear stability analysis ($\epsilon \ll d_e$)

Small-amplitude expansion ($|G| \sim \epsilon \ll d_e$) around equilibrium state

\[
L(\tilde{\phi}, \tilde{\psi}_e) = L(\psi_e) + L^{(1)}(\psi_e; G) + \frac{1}{2} L^{(2)}(\psi_e; G, G) + \frac{1}{6} L^{(3)}(\psi_e; G, G, G) + \ldots
\]

0 at equilibrium

- The 2nd-order Lagrangian $L^{(2)}$ governs the linearized dynamics.

⇒ By putting $G(x, y, t) = \epsilon(t) \hat{x}(x) \frac{\sin ky}{k}$ with $\epsilon(t) \propto e^{\gamma t}$, we obtain

Eigenvalue problem (4th order ODE)

\[
- \left\{ \left[ \left( \gamma/k \right)^2 + (\psi_e')^2 \right] \hat{x}' \right\}' + k^2 \left[ \left( \gamma/k \right)^2 + (\psi_e')^2 \right] \hat{x} = d_e^2 \psi_e' J''' \hat{x} + \psi_e' d_e^2 \nabla^2 (1 - d_e^2 \nabla^2)^{-1} \nabla^2 (\psi_e' \hat{x})
\]

(the prime ' denotes $x$ derivative.)

- Around marginal stability $\gamma \sim 0$, the boundary layers exist at positions where $\psi_e' = 0$.

For $\gamma \neq 0$ and $d_e \neq 0$, the eigenfunctions $\hat{x}$ must be regular.

(The MHD singularity is removed by the electron inertia.)
Energy principle for linear stability

\[-\gamma^2 \delta I = \delta W \] (≡ Eigenvalue problem)

\[\delta I = \int \, dx \frac{1}{k^2} \left( |\hat{\xi}'|^2 + k^2 |\hat{\xi}|^2 \right) > 0\]

\[\delta W = \int \, dx \left[ |\nabla (\psi' \hat{\xi})|^2 + \psi' \psi''' |\hat{\xi}|^2 - \nabla^2 (\psi' \hat{\xi}^*) d_e^2 (1 - d_e^2 \nabla^2)^{-1} \nabla^2 (\psi' \hat{\xi}) \right]\]

> 0 \quad < 0 \quad < 0

(i) magnetic field tension (ii) magnetic shear (iii) electron inertia

• (i)+(ii) > 0 \quad \Rightarrow \quad \text{Stable } \delta W > 0 \text{ in the MHD limit } d_e = 0

• (i)+(iii) > 0 \quad \Rightarrow \quad \text{Stable } \delta W > 0 \text{ without the magnetic shear (or current)}
  
  (The effect of electron inertia weakens the magnetic field tension only in the small scale } \sim d_e)
Test function that makes $\delta W$ negative

- Let us choose the following piecewise-linear function.

\[
\xi(x) = \begin{cases} 
1 & -L_x/2 \leq x < 0 \\
0 & 0 \leq x < 2d_e \\
-0.5 & 2d_e \leq x \leq L_x/2 
\end{cases}
\]

\[
\phi(x, y, t) = \epsilon(t) \xi(x) \sin ky
\]

- Then, the 2nd-order Lagrangian is reduced to

\[
L^{(2)}(\dot{\epsilon}) \simeq \frac{2\pi}{k} B'_y d_e^3 \left[ \left( \frac{d\dot{\epsilon}}{dt} \right)^2 - U(\dot{\epsilon}) \right]
\]

where

\[
\begin{align*}
\dot{\epsilon} &= \epsilon/d_e, & \dot{t} &= t/\tau_e, \\
\tau_e^{-1} &= d_e k B'_y
\end{align*}
\]

Potential energy: $U(\dot{\epsilon}) = -\frac{1+27\epsilon^{-2}}{6} \dot{\epsilon}^2 = -0.776 \dot{\epsilon}^2$

\[
\Rightarrow \text{Linear growth rate: } \gamma = \sqrt{0.776/\tau_e} = 0.881/\tau_e
\]

This agrees with the results of conventional asymptotic matching method as well as our numerical simulation.
Nonlinear stability analysis \((\epsilon > d_e)\)

Remark: Failure of perturbation analysis

Let us try to continue the perturbation expansion of Lagrangian.

Nonlinear perturbations

\[
\tilde{\phi}(x + \partial_y G(x, y, t), y, t) = \partial_t G(x, y, t), \\
\tilde{\psi}_e(x + \partial_y G(x, y, t), y, t) = \psi_e(x)
\]

\[
\Rightarrow \\
\tilde{\phi} = G_t - G_y G_t' + \frac{1}{2} (G_y^2 G_t')' - \frac{1}{6} (G_y^3 G_t')'' + \frac{1}{24} (G_y^4 G_t')''' + O(\epsilon^6), \\
\tilde{\psi}_e = \psi_e - G_y \psi_e' + \frac{1}{2} (G_y^2 \psi_e')' - \frac{1}{6} (G_y^3 \psi_e')'' + \frac{1}{24} (G_y^4 \psi_e')''' + O(\epsilon^5),
\]

where \(G_t = \partial_t G, G_y = \partial_y G\).

However, the linearly unstable mode has a steep gradient, \(G' \sim G/d_e\).

\(\Rightarrow\) The above expansion fails to converge when \(\epsilon = \max |G_y| \to d_e\).

(In fact, we will find that \(\epsilon\) easily exceeds \(d_e\).)

For \(\epsilon > d_e\), full-nonlinear analysis is required around the inner layers.
Potential energy change “around the X-point”

We have directly imposed a nonlinear displacement $\epsilon > d_e$ and investigated subsequent potential energy change.

Around the X point, decrease of potential energy is found to be steeper than that in the linear regime

- Around the X point, $\psi_e$ is compressed by the inflow.
- By this convection, outer region loses magnetic energy of $O(\epsilon^3)$, but inner layer gains magnetic and current energy, at most, of $O(\epsilon^2)$. $\Rightarrow$ Potential decreases in $\epsilon^3$

(When $\epsilon = 5d_e$)

$$
\int d^2_e J_2^2 \, dx = O(\epsilon^2) \\
\int B^2(0) \, dx = \frac{(d_e + \epsilon)^3}{2} + O(\epsilon^2)
$$
Potential energy change in entire domain

As a whole, “smoothness” of the test function is found to be essential for steep decrease of potential energy.

- When the flux $\psi_e$ turns back from the O point side by convection, the potential does not further decrease.
- When the X point elongates and approaches to the “Y-shape”, the potential decreases in cubic power of $\hat{\epsilon}$.
Verification using direct numerical simulation

We calculate the potential energy $U(\hat{\epsilon})$ in the direct numerical simulation.

- Simulation almost agrees with the test function 2 up to $\epsilon < 7d_e$.
- In simulation, flow pattern $\phi$ tends to smooth gradually in time.
  $\Rightarrow$ The Y-shape seems to be self-organized, searching for the lowest $U$ state.
- The nonlinear acceleration force $F(\hat{\epsilon}) = -U'(\hat{\epsilon}) \sim \hat{\epsilon}^2$ is different from $F(\hat{\epsilon}) \sim \hat{\epsilon}^4$ in Ottaviani & Porceli (1993), but simulation agrees with our scaling.
Summary

- We have performed nonlinear analysis and simulation of magnetic reconnection driven by electron inertia, to clarify its acceleration mechanism.

- By formulating variational principle (Lagrangian) of a two-fluid model, growth of magnetic island can be predicted by finding a test function that minimizes potential energy of the system.
  
  - In linear phase \((\epsilon \ll d_e)\), the exponential growth rate \(\epsilon(t) \propto e^{\gamma t}\) is estimated by using a piecewise-linear function that is similar to the eigenfunction. \[\text{Potential } U(\hat{e}^2) = -0.776\hat{e}^2 + O(\hat{e}^3)\]
  
  - In nonlinear phase \((d_e < \epsilon \ll L_x)\), a smooth test function predicts decrease of potential energy \(U \sim -\hat{e}^3\) which is steeper than the linear phase.

  \[\Rightarrow \text{Explosive growth of island } (\epsilon) \text{ during a finite time } \sim \tau_e = (d_e q' \omega A_0)^{-1}\]

  Although the model is too simple at present, this time scale (for large tokamaks, \(\tau_e \sim 100\mu s\)) does not contradict the experimental collapse times.

- By taking a form of Y-shape, most part of magnetic energy flowing into the inner layer is converted into kinetic energy.