Neoclassical Toroidal Viscosity Calculations in Tokamaks using a δf Monte Carlo Simulation

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Outlines (1)

- **Neoclassical toroidal viscosity (NTV)** arises from very small asymmetric magnetic perturbations of \(10^{-2} \sim 10^{-4} \times B_0\) by the error field, MHD activities, or external perturbation coils applied to mitigate ELMs (RMPs).

- NTV is considered to damps plasma toroidal rotation, which may affect the stability of other MHD modes (RWM, locked modes).

- Observed toroidal rotation damping rates in NSTX, JET, etc., have been studied with analytic theories of NTV which are derived from bounce-average drift-kinetic equation [ex. Shaing et al, Nucl. Fusion 2010].

- **Analytic formulae** are usually given in an asymptotic limit which is valid only in a certain range of collisionality and \(E \times B\) rotation speed, or connection formula of such approximated solutions. They also rely on many approximations that are used in conventional neoclassical transport theories.

- It has not been well examined how quantitatively accurate those analytic formulae are.
Outlines (2)

- To develop a simulation scheme for precise and quantitative reliable evaluation of NTV, **a drift–kinetic neoclassical transport code** for helical plasmas, **FORTEC–3D**, is applied to **direct simulation of NTV by the δf Monte Carlo method**. (Satake et al., PPCF 2011, PRL 2011)

- **Benchmark tests for E × B → 0 case** has revealed that the asymptotic analytic formulae overestimates the NTV in low–collisionality regime, while **the combined analytic formula by Park et al. [PRL 2009]** agrees with **FORTEC–3D direct simulation** in wide range of collision frequency.

- We further benchmarked **the NTV calculations in the finite–** E × B **rotation cases**, in low–collisionality tokamak with a single–helicity magnetic perturbation.
  - It is found that the radial profile and the dependence of NTV on $E_r$ show qualitative difference between the $\delta f$ simulation and the analytic formula when $|E_r|$ becomes large.
  - This difference is related to parallel flow shear which develops around the resonant surface in FORTEC–3D simulation.
Basic relations for NTV calculations

● Momentum balance equation
\[
\frac{\partial}{\partial t}(mn\mathbf{u}) = -\nabla \cdot \mathbf{P} + en\mathbf{u} \times \mathbf{B} + en\mathbf{E} + \mathbf{F} + \mathbf{S}_m
\]

\[
\begin{aligned}
\mathbf{F} &= \int d^3v \, m\mathbf{v}C(\delta f) : \text{Friction force} \\
\mathbf{S}_m &= \int d^3v \, m\mathbf{v}S : \text{Momentum input}
\end{aligned}
\]

● Radial particle flux \([\langle \nabla \psi \cdot \left(\frac{\mathbf{b}}{B} \right) \rangle \times \text{above eq.}]\)
\[
\Gamma_{\psi} = \left\langle \frac{\mathbf{F}_\perp \cdot \mathbf{B} \times \nabla \psi}{eB^2} \right\rangle + \left\langle n\mathbf{v}_E \times \mathbf{B} \cdot \nabla \psi \right\rangle - \left\langle \frac{\mathbf{B} \times \nabla \psi \cdot \nabla \cdot \mathbf{P}}{eB^2} \right\rangle + \left\langle \mathbf{S}_{m\perp} \cdot \mathbf{B} \times \nabla \psi \right\rangle + \left\langle \frac{\partial}{\partial t} \left(\frac{n}{\Omega} \mathbf{b} \times \mathbf{u}_\perp \right) \cdot \nabla \psi \right\rangle
\]

\[
\begin{aligned}
\Gamma_{cl} &\quad \text{Classical flux} \\
\Gamma_{E\times B} &\quad \text{Radial } \mathbf{E} \times \mathbf{B} \text{ flux (neglected in FORTEC-3D)} \\
\Gamma_{NC} &\quad \text{Momentum-input driven flux} \\
\Gamma_{pl} &\quad \text{Polarization flux}
\end{aligned}
\]

● Neoclassical flux
\[
\Gamma_{NC} = -\left\langle \frac{\mathbf{B} \times \nabla \psi \cdot \nabla \cdot \mathbf{\hat{P}}}{eB^2} \right\rangle = -\frac{G}{e\iota} \left\langle \frac{\mathbf{B} \cdot \nabla \cdot \mathbf{\hat{P}}}{B^2} \right\rangle + \frac{1}{e\iota} \left\langle \mathbf{e}_\zeta \cdot \nabla \cdot \mathbf{\hat{P}} \right\rangle = \Gamma_{p\parallel} + \Gamma_{TV}
\]

(Banana-plateau & Pfirsch-Schluter fluxes)

● Time evolution of toroidal angular momentum
\[
(\langle \mathbf{e}_\zeta \cdot \rangle \text{ product of the momentum balance eq.})
\]
\[
\sum_a \left\langle \frac{\partial}{\partial t} \mathcal{L}_a^{\zeta} \right\rangle = \sum_a \left( -e_a i\Gamma_{TV}^a + T_\zeta^a \right) + \langle \mathbf{J} \times \mathbf{B} \cdot \mathbf{e}_\zeta \rangle.
\]
\[
\left\langle \mathcal{L}_\zeta \right\rangle \equiv \left\langle mn\mathbf{u} \cdot \mathbf{e}_\zeta \right\rangle
\]


e\iota \Gamma_{TV} = \left\langle \mathbf{e}_\zeta \cdot \nabla \cdot \mathbf{\hat{P}} \right\rangle,

Toroidal rotation is accelerated by the toroidal viscosity \(e\iota \Gamma_{TV}\), external torque input \(T_\zeta\), and the torque of \(\mathbf{J} \times \mathbf{B}\) force.
Evaluation of pressure tensor and NTV in the $\delta f$ method

The guiding-center distribution function: $f = f_M(\psi, v) + \delta f(\psi, \theta, \zeta, v_\parallel, v_\perp)$

\[ \Rightarrow \quad \vec{P} \simeq \vec{P}_{CGL} = p_0(\psi) \vec{I} + \delta P_\parallel \vec{b} + \delta P_\perp (\vec{I} - \vec{b}). \]

Taking an flux surface average $\Rightarrow \langle e_\zeta \cdot \nabla \cdot \vec{P} \rangle = \frac{1}{2} \langle \frac{\partial}{\partial \zeta} \delta P \rangle,$

where $\delta P = \delta P_\parallel + \delta P_\perp = m \int d^3v (v_\parallel^2/2 + v_\perp^2) \delta f.$

Making use of the magnetic field expression in Fourier series in Boozer coordinates, NTV is calculated as follows:

\[ B(\psi, \theta, \zeta) = B_0 \left[ 1 - \sum_{m \geq 1} \epsilon_m(\psi) \cos(m\theta) + \sum_{m \geq 0, n \neq 0} \delta_{m,n}(\psi) \cos(m\theta - n\zeta) \right], \]

\[ \frac{1}{2} \left\langle \frac{\partial}{\partial \zeta} \delta P \right\rangle = \frac{G + \iota I}{2V'} \int d\theta d\zeta \frac{1}{B^2} \frac{\partial \delta P}{\partial \zeta} = \frac{G + \iota I}{V'} B_0 \int d\theta d\zeta \frac{\delta P}{B^3} \sum_{m,n} n \delta_{m,n} \sin(m\theta - n\zeta). \]

Then, the toroidal viscosity is evaluated by decomposing into the following form:

\[ \langle e_\zeta \cdot \nabla \cdot \vec{P} \rangle = \sum_{m,n} \langle e_\zeta \cdot \nabla \cdot \vec{P} \rangle_{m,n} = B_0 \sum_{m,n} n \delta_{m,n} Q_{m,n}. \]

\[ Q_{m,n} \equiv \left\langle \frac{\delta P}{B} \sin(m\theta - n\zeta) \right\rangle. \]

In this expression, one needs to evaluate only the $Q_{m,n}$ components which has corresponding non-zero $\delta_{m,n}$ ($n \neq 0$) perturbations applied.
Basic properties of neoclassical toroidal viscosity
Definition of asymptotic collisionality regimes

- Collisionality dependence is checked for \( E_r = 0 \) cases.
- Parameter survey is carried out by magnifying the collisionality in the range SBP to plateau regimes.
- Single-helicity perturbation, \( \delta_{7,3} = 0.02 (r/a)^2 \cos(7\theta - 3\zeta) \) is superimposed on cylindrical tokamak field, which has a resonant surface at \( r \approx 0.49a \) where \( q=7/3 \).
Combined NTV theory (J. K. Park)

• Including the missing components in conventional bounce-average theories:
  – Resonance between bounce motions and electric precession
  – Resonance between magnetic and electric precession (SBP and SB)
• Do not use assumptions that limits the range of collisionality (such as $1/\nu$, superbanana, etc.)
• Combined formula for NTV torque has been derived with effective Krook collision operator

\[
\langle \hat{\phi} \cdot \nabla \cdot \overline{\Pi}_a \rangle_\ell &= \frac{\epsilon^{-1/2}p_a}{\sqrt{2\pi^{3/2}} R_0} \int_0^1 dk^2 \delta_{w,\ell}^2 \int_0^\infty dx R_{a1\ell} \left[ u^\varphi + 2.0\sigma \left| \frac{1}{e} \frac{dT_a}{d\chi} \right| \right] \\
\tau(\text{Transport}) &\propto \sum_{m,m'} \delta_{m,n} \delta_{m',n} \text{ Resonance } \text{ Rotation with offset}
\]

\[
R_{a y \ell} = \frac{1}{2} \frac{n^2(1 + (\ell^2)^2)\nu_a}{\omega_{ta} \sqrt{x} - n\omega_E - n\sigma \frac{n^3}{4\epsilon} (\omega_{ta}/\omega_{ga}) x}^2 + \left[ 1 + (\ell^2)^2 \right]^2 x^{-3}.
\]

[Park et al, PRL102, 065002 (2009)]
Benchmark test : (m,n)=(7,3) single-mode perturbation, $E_r = 0$ limit

Radial profile of neoclassical toroidal viscosity

- The overall tendency of radial profile is similar. Both results have a strong peak of NTV at the resonant flux surface (q=7/3).
- The peak values are close (within factor 2~3) between FORTEC-3D and the combined formula, but FORTEC-3D has larger NTV tail values at the off-resonant tails.
- Shaing’s $1/\nu$ formula for $\nu_* \approx 0.1$ is also compared here. The peak value is much larger, while it drops rapidly towards the off-resonant surfaces.
Benchmark test: \((m,n)=(7,3)\) single-mode perturbation, \(E_r = 0\) limit

NTV Dependence on collision frequency

Collisionality dependence: (a) Single mode

- No clear \(1/\nu\), or Superbanana-plateau type dependence are found in the FORTEC-3D simulation.
- Asymptotic limit formula is valid only in a narrow range \(\nu_* \sim 1\).
- Good agreement with Park’s combined formula is found over the wide range of \(\nu_*\), in the \(E_r = 0\) limit.
Benchmark of NTV in perturbed tokamaks: **Finite-$E_r$ case**

How is the force balance relation used in NC transport theory?

Example: in Park’s formula ($\chi =$ poloidal flux)

\[
\langle e_\zeta \cdot \nabla \cdot \Pi \rangle \simeq - \frac{\sqrt{epT}}{2e\pi^{3/2} R_0} \sum_{nmm'} \int_0^1 dk^2 \delta_{i,mn'}^2 n(k) \int_0^\infty dx \left[ A_1 + (x - 5/2)A_2 \right] R_{1l}(x),
\]

\[
A_1 = \frac{d \ln p}{d \chi} + \frac{e}{T} \frac{d \Phi}{d \chi}, \quad A_2 = \frac{d \ln T}{d \chi}
\]

$\leftarrow$ "Driving forces" of NTV

**Radial Force balance relation** in torus plasmas: $A_1 = \frac{e}{T} (q u^\theta - u^\zeta)$.

Impose incompressibility on the flow tangent to a flux surface: $\nabla \cdot (n u_\parallel + n u_\perp) = 0$

\[
\Rightarrow n u_\parallel = - \frac{I}{eB} \left( \frac{dp}{d\chi} + en \frac{d \Phi}{d \chi} \right) + K(\chi) B
\]

From neoclassical theory: $K(\chi) = k \frac{I_p}{e \langle B^2 \rangle} A_2$, $n u^\theta = KB \cdot \nabla \theta$

\[
\Rightarrow u^\zeta = - \frac{T}{e} (A_1 - k A_2) / A_2 : \text{Neoclassical offset rotation}
\]

NTV can finally be rewritten in terms of $u_i^\zeta$:

\[
\langle e_\zeta \cdot \nabla \cdot \Pi \rangle \simeq - \frac{\sqrt{ep u_i^\zeta}}{2e\pi^{3/2} R_0} \sum_{nmm'} \int_0^1 dk^2 \delta_{i,mn'}^2 n(k) \int_0^\infty dx R_{1l}(x),
\]

where $u_i^\zeta \equiv u^\zeta - \frac{T}{e} \left[ k + \frac{1}{A_2} \int \frac{d x R_{2l}(x)}{d x R_{1l}(x)} \right] A_2$, $R_{2l}(x) \equiv (x - 5/2)R_{1l}(x)$.

$\leftarrow u^\zeta$ is treated as a driving force here.
Benchmark of NTV in perturbed tokamaks: Finite-$E_r$ case

How are simulations with $E \times B$ rotation compared with theory?

$$u_i^e \equiv u_i^s - \frac{T}{e} \left( k_1 + \frac{\int dx R_{21}(x)}{\int dx R_{14}(x)} \right) A_2.$$ 

- In order to avoid ambiguity coming from approximating $k(\epsilon_t, \nu_*) \approx 1$ used in the most analytic formulae ($k \approx 1.17$ in the $\epsilon_t, \nu_* \to 0$ limit), benchmarks are carried out by setting $A_2 \propto dT/dr = 0$.

- In FORTEC-3D, an $E_r$ profile is given from the force balance relation with assuming $u_\parallel \approx 0$, e.g., $E_r = \frac{T_i}{e} \frac{d}{dx} \ln n_i$. This $E_r$ profile is called “the $E_{r0}$ case”.

- To see the NTV dependence on $E_r$, the $E_{r0}$ case profile is multiplied by ±0.5, 1.0, etc.

- The collisionality is $\nu_* \approx 0.06(@r/a = 0.49)$, around which the NTV becomes maximum in the $E_r \to 0$ limit.

Figure: The radial profiles of $\nu_*$ and $E_{r0}$ (the force-balance profile of $E_r$).
Benchmark test : \((m,n)=(7,3)\) single-mode perturbation, \(\text{Finite-}\mathcal{E}_r\).

Comparison of the NTV radial profile (1) Small-\(|\mathcal{E}_r/\mathcal{E}_r0|\)

- When \(\mathcal{E}_r/\mathcal{E}_r0\) is small, single peak appears at the \(q=7/3\) surface as in \(\mathcal{E}_r = 0\) case. In the force-balance case (\(\mathcal{E}_r = \mathcal{E}_r0\)), NTV vanishes everywhere.

- The discrepancy b/w FORTEC-3D and the analytic formula at the resonant flux surface becomes larger as \(|\mathcal{E}_r/\mathcal{E}_r0|\) increases.
Benchmark test : (m,n)=(7,3) single-mode perturbation, Finite-$E_r$
Comparison of the NTV radial profile (2) Large-$|E_r/E_{r0}|$

- $E_r/E_{r0} = 0$
- $E_r/E_{r0} = -2$
- $E_r/E_{r0} = -10$
- $E_r/E_{r0} = 0.5$
- $E_r/E_{r0} = 2$
- $E_r/E_{r0} = +10$

- Good agreement when $|E_r/E_{r0}| < 1$.  
- At the outside region ($r/a>0.6$) two calculations agrees within a factor $O(1)$ when $E_r/E_{r0} < 0$.  
- Twin peaks appears only in FORTEC-3D. Peaks at the $q=7/3$ resonant disappears in the analytic formula.  
- Large viscosity at $r/a>0.5$ is estimated only in the analytic formula, when $E_r/E_{r0}$ is large positive (negative-$E_r$).
Though \( u_{\parallel} \) evolves slowly in simulation, it remains small, \( u_{\parallel}/v_{th} \ll 0.1 \), in the present benchmark cases.

The evolution of \( u_{\parallel} \) did not seem to affect the evaluation of NTV in the simulations even for large \( E_r \) cases, however …
Though $u_{||}$ evolves slowly in simulation, it remains small, $u_{||}/v_{th} \ll 0.1$, in the present benchmark cases.

The evolution of $u_{||}$ did not seem to affect the evaluation of NTV in the simulations even for large $E_r$ cases, however ...

We found that the $u_{||}$ profiles tend to have shear around the resonant rational flux surface when $E \times B$ rotation is large.

Parallel flow shear viscosity, $\nabla \cdot (m n u_{||} u_{||})$ is a candidate to explain the twin-peak profile of NTV.
Benchmark test : (m,n)=(7,3) single-mode perturbation, Finite-$E_r$

NTV Dependence on the $E_r$ amplitude (2) Total toroidal torque

- The total toroidal torque (NTV integrated on the whole volume) is also compared.
- Since the analytic formula predicts larger NTV on the outer region of plasma when $E_r/E_{r0} \gg 1$, large discrepancy appears when integrated in the plasma volume.

- Why combined analytic formula result is asymmetric in $E_r$ ?

Too much approximation for precession drift (frequency and the direction) ?

\[
\langle \hat{\phi} \cdot \nabla \cdot \Pi_a \rangle_\ell = \frac{\epsilon^{-1/2} \rho_a}{\sqrt{2} \pi^{3/2} R_0} \int_0^1 dk^2 \delta_{w,\ell}^2 \int_0^\infty dx R_{a1\ell} \left[ u^\varphi + 2.0\sigma \left( \frac{1}{e} \frac{dT_a}{d\chi} \right) \right]
\]
\[
R_{a\ell} = \frac{1}{2} \frac{n^2 (1 + (\frac{\ell}{2})^2) \nu_a x (x - \frac{5}{2}) e^{-x}}{\ell \pi \sqrt{\epsilon} \omega_{t\alpha} \sqrt{x} - n \omega_E - n \sigma \frac{3}{4} \left( \omega_{t\alpha}/\omega_{ga} \right) x} + \left( 1 + (\frac{\ell}{2})^2 \nu_a \right)^2 x^{-3}.
\]

In calculation, a rough approx. for $\omega_t$ is adopted. Direction of the precession is prescribed by $\sigma$, but it may be wrong when $E \times B$ rotation is very fast.
Consideration of viscosity from flow shear (1)

Definition of viscosity
Momentum balance equation: \[ \frac{\partial}{\partial t}(mn\textbf{u}) = -\nabla \cdot \overarrow{\textbf{P}} + en\textbf{u} \times \textbf{B} + en\textbf{E} + \mathbf{F} + S_m \]

FORTEC-3D: Evaluate the covariant product of it \((e_\zeta = \partial x / \partial \zeta)\) in Boozer coordinates
\[ \left\langle \frac{\partial}{\partial t}(mne_\zeta \cdot \textbf{u}) \right\rangle = -\left\langle e_\zeta \cdot \nabla \cdot \overrightarrow{\textbf{P}} \right\rangle + \langle \textbf{J} \times \textbf{B} \cdot e_\zeta \rangle + T_\zeta \]
\[ \overrightarrow{\textbf{P}} \simeq \overrightarrow{\textbf{P}}_{CGL} = p_0(\psi) \overrightarrow{\mathbf{I}} + \delta P_{||} \mathbf{b}b + \delta P_{\perp}(\overrightarrow{\mathbf{I}} - \mathbf{b}b) \]

In Analytic formula: Evaluate \(B_t \cdot \text{product of it}\) in Hamada coordinates; use viscous tensor instead of pressure tensor
\[ p \text{ and } \Pi \text{ are defined on the frame moving with the mean flow } \textbf{u}. \]
\[ \Pi = \overrightarrow{\textbf{P}} - p \overrightarrow{\mathbf{I}} - mn\textbf{uu} \]

Momentum balance equation expressed in somewhat different way \((B_t = \psi' \nabla V \times \nabla \theta)\)
\[ \left\langle mn \frac{\partial}{\partial t}(B_t \cdot \textbf{u}) + mnB_t \cdot \textbf{u} \cdot \nabla \textbf{u} \right\rangle = -\left\langle B_t \cdot \nabla \cdot \overrightarrow{\Pi} \right\rangle + \langle \textbf{J} \times \textbf{B} \cdot B_t \rangle + T_\zeta \]

Be careful about the position of the partial derivative
This convective derivative term is usually neglected.
Consideration of viscosity from flow shear (2)

Viscosity tensor when the mean flow is not small:
\[ \nabla \cdot \vec{\Pi} = \nabla \cdot \vec{\Pi}_{CGL} + \nabla \cdot (mn\vec{u}\vec{u}) + (\text{others}) \]

FORTEC-3D:
\[ \simeq \nabla p_0 + \nabla \cdot (\delta\vec{P}_{\parallel} + \delta\vec{P}_{\perp}) + \nabla \cdot [mn(\vec{u}_{\parallel}\vec{u}_E + \vec{u}_E\vec{u}_{\parallel} + \vec{u}_E\vec{u}_E)] \]
\[ (\vec{u}_E = \vec{E} \times \vec{B}/B^2) \]

Effect of parallel flow shear (\(mn\vec{u}_{\parallel}\vec{u}_{\parallel}\)) is included here in the \(\delta f\) simulation.

ExB flow shear viscosity is not considered in present FORTEC-3D simulation.

May affect when \(M_p \sim 1\) or \(u_{\parallel}/v_{th} \sim 1\).

Analytic formula:
\[ \nabla \cdot \vec{\Pi} = \nabla \cdot (\vec{P} - p\vec{I} - mn\vec{u}\vec{u}) \quad \text{where} \quad \vec{u} = \vec{u}_{\parallel} + \vec{u}_E \]

Since the viscosity tensor is defined on the moving frame, large flow effect is excluded from \(\Pi\). However, the effect of convective derivative term \(mn\vec{u} \cdot \nabla \vec{u}\) is omitted from the momentum balance equation!

In both models, flow shear viscosity should be taken into account when \(u/v_{th} \ll 1, M_p \ll 1\) approximations are not valid.
Summary

- FORTEC-3D code was applied to calculate neoclassical toroidal viscosity in tokamak plasmas with asymmetric magnetic field.
- Benchmark tests were carried out with the combined analytic formula.

In $E_r = 0$ limit:
- Strong peak profile of NTV at the resonant flux surface.
- Smaller NTV compared with asymptotic analytic theories.
- Good agreement with Park’s combined analytic formula.

In finite-$E_r$ cases:
- As $|E_r/E_{r0}|$ increases, absolute value of NTV increases in both calculations both in FORTEC-3D and the combined formula.
- Radial profile of NTV shows significant difference b/w two calculations when $E_r/E_{r0} \gg 1$.
  - In FORTEC-3D simulations, a twin-peak profile of NTV appears around the resonant rational flux surface as $|E_r/E_{r0}|$ increases.
- Parallel flow shear in FORTEC-3D seems to affect the NTV when $|E_r|$ is large.

Future Tasks

- Find the physical reason for the formation of the twin-peak NTV profile when $|E_r|$ is large.
  - What is missing in the combined analytic formula?
  - Why sheared parallel flow develops in FORTEC-3D simulation?
  - What happens if multi-helicity perturbations are applied?
- Investigate the collisionality dependence of NTV when $E_r \neq 0$.
  - Is NTV really scales as $\nu \sim \sqrt{\nu}$ as analytic theory predicts?
- Is finite-orbit-width effects (included only in the $\delta f$ simulation) on NTV important?